

Noisy Directed Search

Ziqi Qiao *

University of Bern

October 2026

Abstract

This paper bridges noisy and directed search in a price-posting game with imperfect sampling (search) and uncoordinated meetings (matching). Sellers are capacity-constrained, and the prospect of rationing deters undercutting, sustaining the law of one price under substantial search friction. When search friction is severe, equilibrium prices are dispersed, but unlike in noisy search, the price distribution can have atoms and gaps; in the large-market limit it is in closed form. The induced price dispersion spreads buyers unevenly across sellers and reduces trade, so the equilibrium is constrained inefficient. This inefficiency is absent when search and matching are modeled in isolation or merged into a single matching function.

Keywords: Search and matching, rationing, price dispersion, market efficiency.

*Email: ziqi.qiao@unibe.ch; I thank Lukas Altermatt, Jean-Michel Benkert, Zach Bethune, Benoit Julien, John Kennan, Rasmus Lentz, Peter Norman, Guillaume Rocheteau, Enchuan Shao, Harald Uhlig, Venky Venkateswaran, Randall Wright, and Fengfan Xiang for helpful comments.

1 Introduction

Two departures from the Walrasian benchmark are pervasive: identical goods sell at different prices,¹ and unmet demand and supply coexist.² Search theory accounts for them on two parallel tracks: noisy and directed search. Each track explains one departure and assumes away the other.

In noisy search (Butters, 1977; Varian, 1980; Burdett and Judd, 1983), buyers sample prices imperfectly (search), so sellers trade off a higher price against a lower volume of sales, which delivers price dispersion. However, sellers have unconstrained capacity, so the model does not deliver rationing, and market tightness plays no role. In directed search (Montgomery, 1991; Peters, 1991; Burdett, Shi, and Wright, 2001), sellers are capacity-constrained, so buyers internalize the risk of being rationed when deciding whom to visit (matching), which delivers rationing and a role for tightness. However, buyers observe all prices, so the model does not deliver price dispersion.

This paper shows that the separation of search and matching is not innocuous. The two frictions act on each other: sampling determines where a buyer is able to trade and hence the incidence of rationing, while the prospect of rationing determines the prices sellers are willing to post. I characterize equilibrium when both frictions are present. Price dispersion, universal under noisy search, is absent unless search friction is severe. The law of one price, universal under directed search, breaks down, but not into the atomless interval of noisy search: the lowest price is an atom, followed by a gap. And the market becomes constrained inefficient—it fails to maximize trade given both frictions—exactly when prices are dispersed, whereas each track in isolation is constrained efficient (Table 1).

I embed imperfect sampling into the price-posting game of Burdett, Shi, and Wright (2001, hereafter BSW). There are $m \geq 2$ homogeneous sellers and $n \geq 2$ homogeneous buyers. Sellers post prices subject to unit capacity; buyers sample each seller independently with probability $\alpha \in (0, 1]$ and choose whom to visit within their samples. A seller visited by several buyers serves one at random. I study symmetric equilibria of a finite-agent market and two large-market limits.

¹See Stigler (1961); Pratt, Wise, and Zeckhauser (1979); Sorensen (2000); Baye, Morgan, and Scholten (2004); Kaplan and Menzio (2015); Kaplan, Menzio, Rudanko, and Trachter (2019).

²See Elsby, Michaels, and Ratner (2015) for a survey of the Beveridge curve. In addition, see Rekkas, Jiang, Wright, and Zhu (2025) for housing, Cohen, Kargar, Lester, and Weill (2024) for corporate bonds, and Campo, Gijsbrechts, and Nisol (2003) for goods.

Table 1: Model comparison

Model	Price	Allocation	Efficient?
Walrasian	Degenerate	Clearing	Yes ¹
Noisy search	Dispersed	Clearing	Yes ²
Directed search	Degenerate	Rationing	Yes ³
Noisy directed search	Dispersed ⁴	Rationing	No ⁴

¹ Assuming no market power or other frictions.

² Constrained efficient given search friction; first-best efficient if every buyer samples at least one seller.

³ Constrained efficient given matching friction.

⁴ Dispersed and constrained inefficient when search friction is severe.

In the finite market, the prospect of rationing counteracts the classical incentive to undercut: there is a cutoff $\bar{\alpha} \in (0, 1)$, strictly decreasing in m and in n , such that a pure-strategy equilibrium exists if and only if $\alpha \geq \bar{\alpha}$. Below the cutoff, every equilibrium has three properties: its support accumulates at the monopoly price, which carries no mass; every seller earns the local-monopoly payoff; and the lowest price in the support is an atom, followed by a gap. The reason is economic: because buyers weigh price against the queue a price attracts, a seller's sales are continuous in price, and a marginal cut buys too little traffic to pay for itself, ruling out continuous undercutting. The characterization is not routine: the dispersed equilibria are neither finite nor atomless, so the standard arguments of either track do not apply.

In the large market, I hold the buyer–seller ratio fixed and distinguish two cases by the average sample size αm . In Case 1, αm grows without bound, as it does for any fixed α , however small, and the BSW equilibrium survives. Observing a random share of prices, rather than all of them, thus suffices for the directed-search allocation. This weakens the informational assumption behind directed search and complements its large-market foundations (Peters, 2000; Galenianos and Kircher, 2012).

In Case 2, αm remains finite in the limit. Unlike the finite market, where no general solution is available, this limit always admits a closed-form equilibrium. As search friction rises, the price support passes through three regimes: a single atom; an atom, a gap, and an interval; and an interval alone. The hybrid regime is the signature of the two tracks acting together: the atom is the directed-search force, and the interval the noisy-search force.

Both tracks are constrained efficient: given its own friction, each maximizes the

volume of trade. Together they are not, once prices are dispersed. Price dispersion spreads buyers unevenly across sellers; because the matching technology is strictly concave in the queue, this queue dispersion reduces aggregate trade below the constrained optimum. This efficiency loss is invisible to macroeconomic models that collapse search and matching into an aggregate matching function, which mechanically imposes uniform queues.³

Decomposing the shortfall of equilibrium trade from the first best shows that the interaction between search and matching can account for more than half the total loss in absolute value. The interaction also changes sign with market tightness: the two frictions destroy surplus as substitutes when buyers are the short side, and can act as complements when they are the long side.

Price dispersion itself is an imprecise measure of efficiency loss. It marks constrained inefficiency exactly, but it is not necessary for a loss relative to the first best, since search friction already reduces trade where the law of one price holds. Moreover, greater dispersion can accompany either a larger or a smaller loss.

This paper is most closely related to Lester (2011) and Huang (2016), who also combine imperfect sampling with limited capacity, but under all-or-nothing information: informed buyers observe all prices and uninformed buyers observe none. Lester (2011) studies how the price level responds to the share of informed buyers in the pure-strategy regime, and Huang (2016) characterizes the mixed-strategy regime with two sellers when that share is small; neither studies efficiency. I characterize all three price regimes in a large market and study their efficiency.

Two papers obtain price distributions of the same three forms. Bethune, Choi, and Wright (2020) obtain them with an exogenous matching function, in a model built for different questions. Menzio (2024) obtains them with increasing marginal costs rather than limited capacity, so rationing plays no role, and shows that price dispersion is inefficient because sellers produce different quantities. Here, matching arises endogenously from buyers' visiting decisions, and the inefficiency works through uneven queues rather than through costs.

More broadly, the model belongs to the class of pricing games without pure-strategy equilibria, where a seller's sales jump when it undercuts a rival, because

³See Rogerson, Shimer, and Wright (2005) for a survey of labor-market applications, Lagos, Rocheteau, and Wright (2017) for a survey of monetary applications, and Wright, Kircher, Julien, and Guerrieri (2021) for a survey of directed and competitive search.

of location preferences (Shilony, 1977), capacity constraints (Peters, 1984), brand loyalty (Narasimhan, 1988), or adverse selection (Broecker, 1990). The mechanism here is the opposite: a seller priced above its rivals still shares their buyers until its price is discretely higher, so its sales are continuous in price. The continuity produces atoms and gaps instead of the usual atomless interval.

The rest of the paper is organized as follows. Section 2 describes the environment and characterizes equilibrium in the finite-agent market. Section 3 examines the large-market limits. Section 4 discusses the efficiency implications and their relation to price dispersion. Section 5 concludes.

2 The finite-agent market

Environment. Consider a market with $n \geq 2$ buyers and $m \geq 2$ sellers, indexed by $j \in \mathcal{S} \equiv \{1, \dots, m\}$. Each seller has one indivisible good, which it values at zero, and each buyer wants one unit, which it values at one. A trade at price $p \in [0, 1]$ gives the seller p and the buyer $1 - p$, and agents who do not trade get zero.

The exchange proceeds in two stages. In the first stage, the sellers simultaneously post prices $\mathbf{p} \equiv (p_1, \dots, p_m)$. The buyers do not observe $\mathbf{p}$ itself, only the multi-set $\{\{p_1, \dots, p_m\}\}$, that is, the empirical distribution of posted prices without the identities of the sellers posting them.⁴

In the second stage, each buyer attempts to buy from one seller, subject to the *search friction*: it samples seller j with probability $\alpha \in (0, 1]$, independently across buyer–seller pairs. A lower α means more severe search friction, and BSW is the special case $\alpha = 1$ with no search friction.

Let $S \subseteq \mathcal{S}$ be the set of sellers a buyer samples and $\mathbf{p}_S \equiv (p_k)_{k \in S}$ the associated prices, both privately known to the buyer. A buyer draws S with probability $\alpha^{|S|}(1 - \alpha)^{m - |S|}$ and may visit seller j only if $j \in S$. In particular, a buyer is *idle*, sampling no seller, with probability $(1 - \alpha)^m$, and is *captive*, sampling exactly one seller, with probability $m\alpha(1 - \alpha)^{m-1}$.

After the visit decisions, each seller j allocates the good uniformly at random to one of the buyers who show up, if any, and the chosen buyer pays the posted

⁴This assumption dispenses with off-path beliefs and is used only for the properties of mixed-strategy equilibria in Proposition 1; the other results, including the large-market limits, also hold under passive beliefs.

price p_j . As in BSW, buyers cannot coordinate their visits, which creates *matching friction*: with positive probability, some buyers are rationed while some sellers have no customers, even when $\alpha = 1$. I now solve the game backward.

Stage 2. Following the literature, I focus on symmetric equilibria in which agents holding the same information play the same strategy. Specifically, buyers use a profile $\sigma \equiv (\sigma_1, \dots, \sigma_m)$, where $\sigma_j(\mathbf{p}_S)$ is the probability that a buyer with sample S and prices $\mathbf{p}_S$ visits seller j , subject to $\sigma_j(\mathbf{p}_S) = 0$ for every $j \notin S$. I adopt the convention that a buyer visits when being indifferent between visiting and staying idle, so $\sum_{j \in S} \sigma_j(\mathbf{p}_S) = 1$ for every $S \neq \emptyset$.

Given σ , the unconditional probability θ_j that a buyer visits seller j is

$$\theta_j = \sum_{S \subseteq \mathcal{S}} \alpha^{|S|} (1 - \alpha)^{m - |S|} \sigma_j(\mathbf{p}_S). \quad (1)$$

I refer to θ_j as seller j 's normalized queue. When serving only its captive buyers, a seller faces the lowest possible queue $\underline{\theta} \equiv \alpha(1 - \alpha)^{m-1}$.

Since each of the other $n - 1$ buyers independently visits seller j with probability θ_j , a buyer who visits j trades with probability

$$\phi(\theta_j) = \sum_{i=0}^{n-1} \binom{n-1}{i} \theta_j^i (1 - \theta_j)^{n-1-i} \frac{1}{i+1} = \frac{1 - (1 - \theta_j)^n}{n\theta_j}, \quad (2)$$

satisfying $\phi(0) = 1$ by continuity, $\phi(1) = 1/n$, and $\phi'(\cdot) < 0$. A buyer's expected payoff from visiting seller j is $(1 - p_j)\phi(\theta_j)$.

Payoffs on both sides depend on σ only through $\theta \equiv (\theta_1, \dots, \theta_m)$. Moreover, a vector θ is induced through (1) by some σ iff it satisfies two conditions. Condition (i), $\sum_{j \in S} \theta_j = 1 - (1 - \alpha)^m$, says that every buyer with a nonempty sample visits a seller. Condition (ii), $\sum_{j \in S} \theta_j \leq 1 - (1 - \alpha)^{|S|}$ for every $S \subseteq \mathcal{S}$, says that the sellers in S attract at most the buyers who sample at least one of them.⁵ The game can therefore be analyzed directly in θ .

Definition 1. Fix $\mathbf{p}$. A buyer equilibrium is a strategy σ^* such that for every $S \neq \emptyset$ and every $j \in S$, $\sigma_j^*(\mathbf{p}_S) > 0$ implies $(1 - p_j)\phi(\theta_j^*) = \max_{k \in S} (1 - p_k)\phi(\theta_k^*)$, where θ^*

⁵Necessity is straightforward. Sufficiency follows from the max-flow min-cut theorem. Condition (ii) is Hall's condition for routing buyers to sellers in their samples so that each seller j receives θ_j , and (i) ensures that every buyer with a nonempty sample is routed.

is induced by σ^* through (1).

Lemma 1. For any $\mathbf{p} \in [0, 1]^m$, a buyer equilibrium exists. Every coordinate θ_j^* with $p_j < 1$ is unique and continuous in $\mathbf{p}$; if at most one seller posts the price 1, the whole vector $\boldsymbol{\theta}^*$ is unique and continuous in $\mathbf{p}$.

Appendix A proves Lemma 1 by showing that the queues induced by buyer equilibria are exactly the maximizers of a concave potential function over the set of $\boldsymbol{\theta}$ satisfying (i) and (ii).

Stage 1. Given $\boldsymbol{\theta}^*$, seller j trades with probability

$$\psi(\theta_j^*) = 1 - (1 - \theta_j^*)^n, \quad (3)$$

which satisfies $\psi'(\cdot) > 0 > \psi''(\cdot)$ on $[0, 1]$. Sellers anticipate how $\boldsymbol{\theta}^*$ responds to $\mathbf{p}$. By Lemma 1, θ_j^* is a continuous function of $\mathbf{p}$ when $p_j < 1$, and a seller's expected payoff is an integral over rivals' prices. Consider a symmetric equilibrium in which sellers randomize according to a distribution F on $[0, 1]$. Seller j 's payoff from posting p_j is

$$V(p_j; F) = p_j \int_{[0, 1]^{m-1}} (\psi \circ \theta_j^*)(p_j, \mathbf{p}_{-j}) dF(\mathbf{p}_{-j}), \quad (4)$$

where $\mathbf{p}_{-j}$ is the vector of prices posted by sellers other than j , and $dF(\mathbf{p}_{-j}) = \prod_{k \neq j} dF(p_k)$ by independence.

Definition 2. An equilibrium is a distribution F^* such that for every $p \in \text{supp}(F^*)$, $V(p; F^*) = \max_{p' \in [0, 1]} V(p'; F^*)$. A pure-strategy (mixed-strategy) equilibrium is an equilibrium with degenerate (nondegenerate) F^* .

The natural candidate is a pure-strategy equilibrium, derived exactly as in Proposition 2 of BSW. Let $\eta(\theta) \equiv \theta\psi'(\theta)/\psi(\theta)$ be the elasticity of a seller's trading probability with respect to its queue, satisfying $0 < \eta(\cdot) < 1$ and $\eta'(\cdot) < 0$ on $(0, 1)$. In the candidate pure-strategy equilibrium, for all $j \in \mathcal{S}$,

$$p_j = p^* = 1 - \frac{(m-1)\eta^*}{m - \eta^*} \text{ with } \theta^* = \frac{1 - (1 - \alpha)^m}{m} \text{ and } \eta^* \equiv \eta(\theta^*) \quad (5)$$

and buyers split evenly across the sellers they sample, $\sigma_j^*(\mathbf{p}_S^*) = 1/|S|$ for $j \in S$. The formula generalizes BSW's, with α entering only through θ^* . The price p^* strictly

increases in α , because lower friction intensifies competition among buyers. As $\alpha \rightarrow 0$, queues vanish and p^* falls to the sellers' reservation value, as in Bertrand competition.

In the candidate equilibrium, all sellers earn the competing payoff $V^C(\alpha) = p^* \psi^*$, where $\psi^* \equiv \psi(\theta^*)$. The relevant deviation is for a seller to serve only its captive buyers, which gives the queue $\underline{\theta}$.⁶ Such a deviation is best made at the monopoly price $p = 1$, which yields the local-monopoly payoff $V^M(\alpha) = \psi(\underline{\theta})$. The candidate is therefore an equilibrium iff $V^C(\alpha) \geq V^M(\alpha)$.

The two payoffs cross exactly once: there is a unique $\bar{\alpha} \in (0, 1)$ such that $V^C(\bar{\alpha}) = V^M(\bar{\alpha})$, and $V^C(\alpha) > V^M(\alpha)$ iff $\alpha > \bar{\alpha}$, with $\bar{\alpha}$ strictly decreasing in m and in n (Appendix C). Moreover, $\bar{\alpha}$ vanishes as $m, n \rightarrow \infty$ with n/m fixed (Section 3). Whether the law of one price can hold in equilibrium therefore follows a cutoff rule in the severity of search friction.

Proposition 1. Fix m and n . If $\alpha \geq \bar{\alpha}$, (5) is a pure-strategy equilibrium. If $\alpha < \bar{\alpha}$, every equilibrium is in mixed strategies, and in every equilibrium (i) the support of F^* accumulates at the monopoly price $p = 1$, which carries no mass; (ii) every seller earns the local-monopoly payoff $V^M(\alpha)$; and (iii) the lowest price in $\text{supp}(F^*)$ is an atom followed by a gap.

For $\alpha \geq \bar{\alpha}$, the proof follows that of Proposition 2 in BSW once the buyers' problem is written in terms of θ ; Appendix B proves the case $\alpha < \bar{\alpha}$.

Three features distinguish the result from noisy search, where any search friction, however small, generates dispersion. First, the law of one price survives substantial search friction: the prospect of rationing already deters undercutting, so dispersion becomes unavoidable only when α falls below $\bar{\alpha}$.

Second, once dispersion emerges, it is not a continuum. Because buyers weigh price against queue, a seller's sales are continuous in its price, and a small price cut gains little. Just above the lowest price in the support, sales are also concave in price, so revenue is strictly concave there, and at most one price can be optimal. Consequently, the lowest price is isolated in the support, hence an atom, and the prices just above it form a gap. At the top, by contrast, the equilibrium resembles noisy search: the support accumulates at the monopoly price, and every seller earns exactly V^M . No seller posts $p = 1$ with positive probability, since sellers tied there would gain by

⁶Deviations after which the seller still shares buyers with its rivals are unprofitable, by the argument of Proposition 2 in BSW.

undercutting, so a seller posting $p = 1$ is almost surely the most expensive, serves only its captive buyers, and earns V^M . The support must therefore reach $p = 1$: if it stopped below one, a seller at the top would, with positive probability, share buyers with rivals, and at its first-order condition it would earn less than V^M in expectation.

Third, price dispersion reduces trade, unlike in noisy search, where every buyer who samples a seller trades. Condition (i) fixes $\sum_j \theta_j^*$, and ψ is strictly concave, so expected trade $\sum_j \psi(\theta_j^*)$ is maximized only by equal queues, which the pure-strategy equilibrium delivers and a planner who cannot coordinate visits could not improve on. By (ii), dispersed prices preclude equal queues; otherwise, a seller's payoff $p\psi^*$ rises with its price. Hence the equilibrium is constrained inefficient exactly when prices are dispersed; Section 4 quantifies the loss in the large market.

Proposition 1 constrains every equilibrium below the cutoff but constructs none; for general (m, n) no construction is available, while the case $m = 2$ yields an explicit one, as in Huang (2016). The support is then an infinite increasing sequence of atoms, which converges to the monopoly price $p = 1$ by Proposition 1. Consecutive atoms are far enough apart that a seller at one atom gains nothing from moving toward the next, so each atom is pinned down by the one below it through an explicit difference equation (Appendix D).

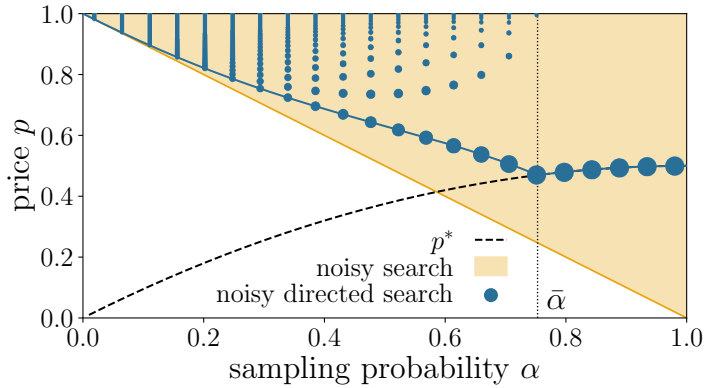


Figure 1: Equilibrium pricing: $m = n = 2$.

Figure 1 plots equilibrium prices against α for $m = n = 2$. Each dot is an atom, with size proportional to its mass, and the solid line connects the lowest atoms. For $\alpha \geq \bar{\alpha} \approx 0.75$ (dotted vertical line), the pure-strategy price p^* in (5) is an equilibrium; the dashed line extends p^* below $\bar{\alpha}$, where it is no longer an equilibrium. For $\alpha < \bar{\alpha}$, the atoms rise toward one. They lie inside the shaded region, the support of the

noisy-search equilibrium in which sellers have unlimited capacity. Intuitively, sellers compete less than they would with unlimited capacity, because buyers take the risk of rationing into account when deciding whom to visit.

3 The large-market limits

Now let the market grow large, $m, n \rightarrow \infty$. It helps to start from the equilibrium queue: if $\theta^* \rightarrow 0$ and $n\theta^* \rightarrow \varepsilon < \infty$, then p^* in (5) and ψ^* satisfy

$$p^* \rightarrow 1 - \frac{\varepsilon}{e^\varepsilon - 1} \quad \text{and} \quad \psi^* \rightarrow 1 - e^{-\varepsilon}, \quad (6)$$

which are the BSW limit price and the urn-ball matching function, with one difference. In place of the tightness $\tau \equiv n/m$, they depend on the effective tightness ε , the expected number of buyers who visit a given seller.

The effective tightness equals τ without search friction ($\alpha = 1$), as in BSW, but $\alpha = 1$ is not necessary. Since $n\theta^* = \tau[1 - (1 - \alpha)^m]$, only buyers who are not idle count toward the effective tightness. Therefore, $\varepsilon = \tau$ whenever the share $(1 - \alpha)^m$ of idle buyers tends to zero. Whether this share vanishes depends on how the mean sample size αm scales as the market grows. Holding τ fixed, I distinguish two cases: $\alpha m \rightarrow \infty$ (Case 1), so that $(1 - \alpha)^m \rightarrow 0$, and $\alpha m \rightarrow \mu < \infty$ (Case 2), so that $(1 - \alpha)^m \rightarrow e^{-\mu}$.

3.1 Case 1: $\alpha m \rightarrow \infty$

In this case, each buyer samples infinitely many sellers in the limit. For simplicity, suppose α is constant, so that each buyer samples a fixed fraction of sellers, or equivalently each seller advertises to a fixed fraction of buyers. The result below requires only $\alpha m \rightarrow \infty$.

Since the idle share vanishes, the effective tightness equals τ for every α , and the candidate pure-strategy equilibrium coincides with that of BSW. A higher α raises the chance that a buyer samples a given seller, but it also raises the number of sellers in the buyer's sample, about αm . The probability of visiting the given seller is thus about $\alpha/(\alpha m) = 1/m$, independent of α .

Proposition 2. Fix τ and α . For all m, n large enough, (5) is an equilibrium, and as $m, n \rightarrow \infty$,

$$p^* \rightarrow 1 - \frac{\tau}{e^\tau - 1} \quad \text{and} \quad \psi^* \rightarrow 1 - e^{-\tau}. \quad (7)$$

In a large market, the pure-strategy candidate is an equilibrium for every α . The mixed-strategy region vanishes because captive buyers do: a seller's expected number of captive buyers, $n\alpha(1-\alpha)^{m-1} \leq \tau\alpha me^{-\alpha(m-1)}$, tends to zero, and so does its local-monopoly payoff V^M . Its competing payoff, by contrast, tends to $1 - (1+\tau)e^{-\tau} > 0$, so $\bar{\alpha} \rightarrow 0$.

In noisy search, losing captive buyers would drive prices down to sellers' reservation value; here, the prospect of rationing sustains the BSW price. Moreover, because the result holds for any $\alpha > 0$, Case 1 relaxes the informational assumption of directed search. Buyers need not observe all prices: observing a random share of them, however small, supports the same allocation.

3.2 Case 2: $\alpha m \rightarrow \mu < \infty$

In this case, each buyer samples only finitely many sellers, μ on average, even in the limit. For simplicity, suppose $\alpha m = \mu$ stays constant as the market grows, as with geographic or attention constraints, so search friction persists in the large market.

Now $(1-\alpha)^m = (1-\mu/m)^m \rightarrow e^{-\mu}$ and $n\theta^* \rightarrow \tau(1-e^{-\mu})$, so the effective tightness becomes $\varepsilon = \tau(1-e^{-\mu})$. The candidate pure-strategy equilibrium satisfies

$$p^* = 1 - \frac{\tau(1-e^{-\mu})}{\exp(\tau - \tau e^{-\mu}) - 1} \quad \text{and} \quad \psi^* = 1 - e^{-\tau(1-e^{-\mu})}. \quad (8)$$

Unlike in Case 1, search friction, now measured by μ , affects the equilibrium allocation. The mixed-strategy region also survives, because the competing payoff

$$V^C(\mu, \tau) = 1 - (1+\varepsilon)e^{-\varepsilon} \quad (9)$$

can fall below the local-monopoly payoff

$$V^M(\mu, \tau) = 1 - \exp(-\tau\mu e^{-\mu}), \quad (10)$$

where $\tau\mu e^{-\mu}$ is the expected number of captive buyers per seller. A cutoff rule again applies (Appendix C): given τ , there is a unique $\bar{\mu}(\tau) > 0$ such that $V^C(\bar{\mu}, \tau) = V^M(\bar{\mu}, \tau)$, and $V^C(\mu, \tau) > V^M(\mu, \tau)$ iff $\mu > \bar{\mu}(\tau)$.

Unlike in the finite market, a mixed-strategy equilibrium is tractable in the large market. Each price p in the support has its own effective tightness ε_p , the expected number of buyers who visit a seller posting p . Such a seller sells with probability $\Psi(p) \equiv 1 - e^{-\varepsilon_p}$, the large-market counterpart of ψ . In the equilibrium below, as in the finite market, the monopoly price $p = 1$ is in the support and attracts only captive buyers, so $\varepsilon_1 = \tau\mu e^{-\mu}$ and every price in the support earns V^M : $p\Psi(p) = V^M$, or

$$\varepsilon_p = -\ln(1 - V^M/p), \quad (11)$$

which is strictly decreasing in p . I use this formula to define ε_p for every $p \in (V^M, 1]$; off the support, it is the tightness at which a seller posting p would earn V^M .

Let $\Phi(x) \equiv (1 - e^{-x})/x$, the large-market counterpart of ϕ , so that a buyer who visits a seller with tightness ε_p trades with probability $\Phi(\varepsilon_p)$. Finally, let $\Gamma_p \equiv \tau\mu/\varepsilon_p$ be the expected number of buyers who sample a seller, $\tau\mu$, relative to the expected number who visit it at price p .

Proposition 3. Fix τ and μ . If $\mu \geq \bar{\mu}$, (8) is a pure-strategy equilibrium. If $\mu < \bar{\mu}$, there is a mixed-strategy equilibrium F^* satisfying

$$F^*(p) = \begin{cases} 0 & \text{if } p \in [0, \hat{p}) \\ \kappa & \text{if } p \in [\hat{p}, \underline{p}) \\ g(p) & \text{if } p \in [\underline{p}, 1] \end{cases} \quad (12)$$

$$g(p) = \mu^{-1} \ln \Gamma_p \quad (13)$$

$$\hat{p} = V^M W_{-1} \left(\frac{V^M - 1}{e} \right) \left[1 - V^M + W_{-1} \left(\frac{V^M - 1}{e} \right) \right]^{-1} \quad (14)$$

$$\kappa = \mu^{-1} [W_0(-\Gamma_{\hat{p}} e^{-\Gamma_{\hat{p}}}) + \Gamma_{\hat{p}}] \quad (15)$$

$$\underline{p} = V^M [1 - \exp(-\tau\mu e^{-\mu\kappa})]^{-1} \quad (16)$$

where W_0 and W_{-1} are the principal and negative branches of the Lambert W function. Moreover, there is a $\hat{\mu} \in (0, \bar{\mu})$ such that $\kappa = 0$ and $\text{supp}(F^*) = [\underline{p}, 1]$ if $\mu \leq \hat{\mu}$, while $\kappa \in (0, 1)$ and $\text{supp}(F^*) = \{\hat{p}\} \cup [\underline{p}, 1]$ with $\hat{p} < \underline{p}$ if $\hat{\mu} < \mu < \bar{\mu}$.

Proof. The case $\mu \geq \bar{\mu}$ follows as in Proposition 1. Let $\mu < \bar{\mu}$; I construct the candidate (12)–(16) and then verify it.

A buyer who visits a seller posting p obtains

$$u(p) = (1-p)\Phi(\varepsilon_p) = -\frac{V^M(1/p-1)}{\ln(1-V^M/p)}. \quad (17)$$

Since $u(p) \rightarrow 0$ as $p \rightarrow V^M$ or $p \rightarrow 1$, u attains an interior maximum at some $\hat{p} \in (V^M, 1)$: because of congestion, the lowest price is not necessarily the buyers' favorite. As u is concave on $(V^M, 1)$, $\hat{p}$ solves $u'(\hat{p}) = 0$, that is, $ze^z = (V^M - 1)/e$ with $z \equiv \ln(1 - V^M/\hat{p}) - 1 < -1$. Since $(V^M - 1)/e \in (-1/e, 0)$ and $z < -1$, the negative branch W_{-1} gives (14).

By concavity, for every $p \in (\hat{p}, 1)$ there is a unique $q \in (V^M, \hat{p})$ with $u(q) = u(p)$, and buyers strictly prefer every price in (q, p) to p . A buyer who samples a seller posting p therefore visits it exactly when the buyer samples no seller posting a price in (q, p) . The number of other sellers in a buyer's sample is Poisson with mean μ , so the number posting prices in (q, p) is Poisson with mean $\mu[F^*(p) - F^*(q)]$, and $\varepsilon_p = \tau\mu e^{-\mu[F^*(p) - F^*(q)]}$. In the candidate, no seller posts below $\hat{p}$, so $F^*(q) = 0$ and $F^*(p) = g(p)$ in (13), with $g(1) = 1$. Without an atom, the interval would start at

$$p_0 \equiv g^{-1}(0) = \frac{V^M}{1 - e^{-\tau\mu}}. \quad (18)$$

Figure 2 shows the two possible positions of p_0 relative to $\hat{p}$. If $p_0 \geq \hat{p}$ (left panel), u is decreasing on $[p_0, 1]$, so buyers always visit the cheapest seller they sample, and the noisy-search construction applies: $\kappa = 0$ and $F^* = g$ on $[p_0, 1]$.

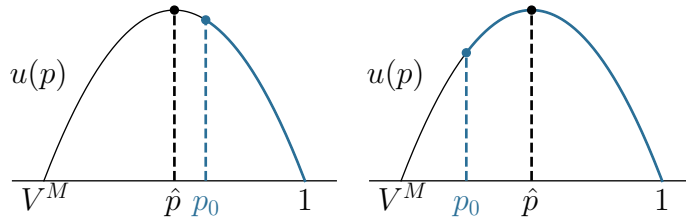


Figure 2: Position of p_0 relative to $\hat{p}$: $p_0 \geq \hat{p}$ (left) and $p_0 < \hat{p}$ (right).

If $p_0 < \hat{p}$ (right panel), buyers prefer $\hat{p}$ to p_0 , so a seller at $\hat{p}$ would attract more buyers than the construction assigns to it, and the lowest price must instead be an

atom at $\hat{p}$ with mass $\kappa > 0$. Since ε_p is strictly decreasing and $\varepsilon_{p_0} = \tau\mu$, the condition $p_0 < \hat{p}$ is equivalent to $\Gamma_{\hat{p}} > 1$, which holds iff $\mu \in (\hat{\mu}, \bar{\mu})$ (Appendix C).

To find κ , note that a buyer who samples an atom seller also samples a Poisson number of other atom sellers, with mean $\mu\kappa$, and visits one of the atom sellers in its sample at random. An atom seller thus receives a share $\Phi(\mu\kappa)$ of the $\tau\mu$ buyers who sample it, and its tightness $\tau\mu\Phi(\mu\kappa)$ must equal $\varepsilon_{\hat{p}}$:

$$h(\kappa) \equiv \tau(1 - e^{-\mu\kappa}) - \kappa\varepsilon_{\hat{p}} = 0. \quad (19)$$

The function h is strictly concave, with $h(0) = 0$, $h'(0) = \varepsilon_{\hat{p}}(\Gamma_{\hat{p}} - 1)$, and $h(\kappa) < 0$ for large κ . Hence $\kappa = 0$ is the only nonnegative root of (19) if $\Gamma_{\hat{p}} \leq 1$, and there is a second root $\kappa \in (0, 1)$ if $\Gamma_{\hat{p}} > 1$, as $\mu < \bar{\mu}$ gives $h(1) = \varepsilon - \varepsilon_{\hat{p}} < 0$ by (C.8) and (C.6). With $y \equiv \mu\kappa - \Gamma_{\hat{p}}$, (19) becomes $ye^y = -\Gamma_{\hat{p}}e^{-\Gamma_{\hat{p}}} \in [-1/e, 0)$. If $\Gamma_{\hat{p}} > 1$, W_0 gives the positive root (15); if $\Gamma_{\hat{p}} \leq 1$, W_0 gives $y = -\Gamma_{\hat{p}}$, so (15) gives $\kappa = 0$.

Given κ , the interval starts at $\underline{p} = g^{-1}(\kappa)$, where $\varepsilon_{\underline{p}} = \tau\mu e^{-\mu\kappa}$; this gives (16). When $\kappa > 0$, $\varepsilon_{\underline{p}} < \tau\mu\Phi(\mu\kappa) = \varepsilon_{\hat{p}}$ because $e^{-x} < \Phi(x)$ for $x > 0$, so $\underline{p} > \hat{p}$: a gap separates the atom from the interval.

It remains to rule out deviations to prices outside the support. A price $p \leq V^M$ earns less than V^M . Consider $p \in (V^M, \underline{p})$ outside the support, and let $\tilde{\varepsilon}$ be the deviator's tightness; the deviation earns $p(1 - e^{-\tilde{\varepsilon}})$, which exceeds $V^M = p(1 - e^{-\varepsilon_p})$ only if $\tilde{\varepsilon} > \varepsilon_p$. Its visitors would then obtain $(1-p)\Phi(\tilde{\varepsilon}) < (1-p)\Phi(\varepsilon_p) = u(p) \leq u(\hat{p})$, so none of them samples an atom seller, and $\tilde{\varepsilon} \leq \tau\mu e^{-\mu\kappa} = \varepsilon_{\underline{p}} < \varepsilon_p$, a contradiction. Therefore, (12) is an equilibrium. $\blacksquare$

Proposition 3 constructs an equilibrium for every μ , and the cutoffs $\hat{\mu} < \bar{\mu}$ define three regimes. When search friction is mild ($\mu \geq \bar{\mu}$), the equilibrium takes the BSW form, a single price. When it is severe ($\mu \leq \hat{\mu}$), the equilibrium takes the noisy-search form, an interval of prices. In between ($\hat{\mu} < \mu < \bar{\mu}$), it is a hybrid of the two, with an atom, a gap, and an interval. Figure 3 plots an example with $(\tau, \mu) = (1, 1.5)$.

The atom and the gap come together. The atom price $\hat{p}$ maximizes buyers' payoff among all price–queue pairs that give sellers V^M , the local Hosios condition (Hosios, 1990) discussed in Section 4. A seller that undercuts $\hat{p}$ attracts buyers away from the atom only until they are indifferent, and the longer queue does not make up for the lower price.

The gap arises because a seller priced just above $\hat{p}$ still competes with the atom

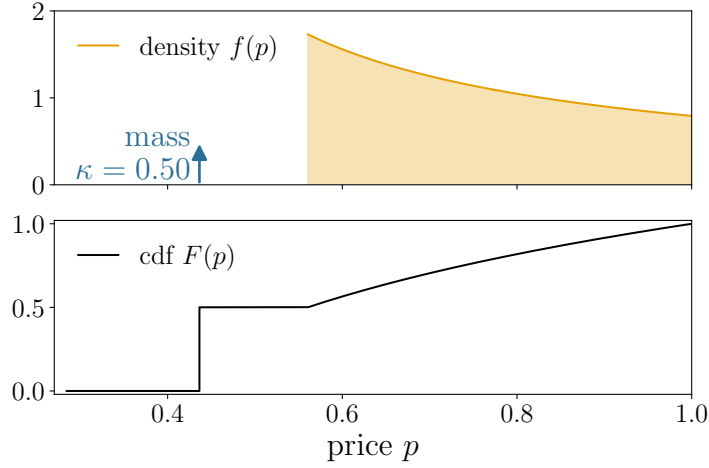


Figure 3: Mixed-strategy equilibrium: $(\tau, \mu) = (1, 1.5)$.

sellers. It keeps the buyers who also sample an atom seller only by leaving them indifferent between it and the atom, so its queue shrinks as its price rises; the shorter queue outweighs the higher price, and, as with undercutting, its payoff falls below V^M . Once its price is high enough, it loses these buyers altogether and serves only those who sample no atom seller. Its queue then stops shrinking, so its payoff rises with its price, but it reaches V^M only at $\underline{p}$.

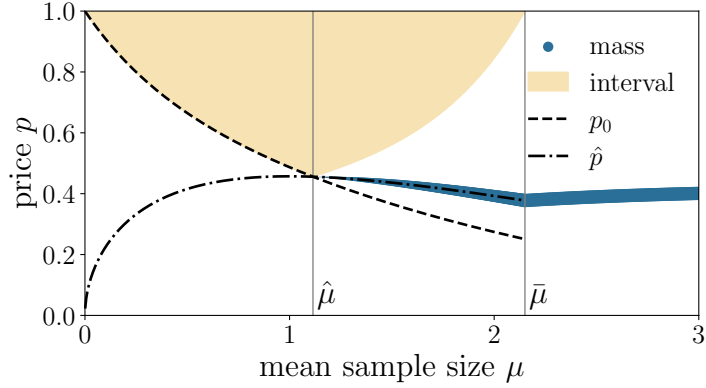


Figure 4: Case 2 equilibrium: $\tau = 1$.

Although the support changes shape across the three regimes, the equilibrium varies continuously with μ . Figure 4 illustrates this for $\tau = 1$. In the middle regime, the atom mass κ rises from 0 at $\hat{\mu}$ to 1 at $\bar{\mu}$, where the atom merges with the pure-strategy price, while the interval shrinks toward $p = 1$. The dashed and dash-dotted lines plot p_0 and $\hat{p}$, which cross at $\hat{\mu}$.

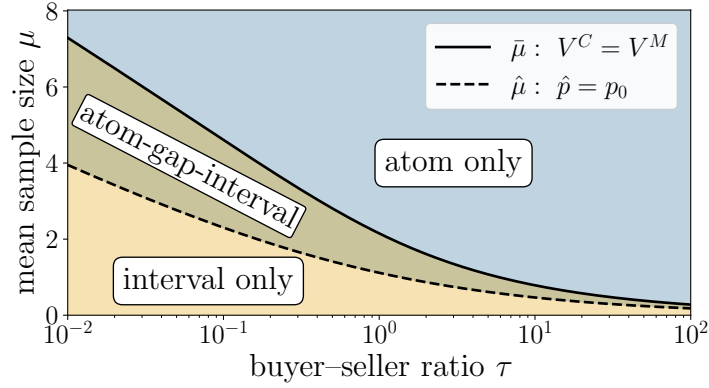


Figure 5: Case 2 equilibrium cutoffs.

Figure 5 plots $\hat{\mu}$ and $\bar{\mu}$ against τ . Blue shading marks regimes with an atom, and yellow shading marks regimes with an interval, so their overlap is the atom-gap-interval regime; later figures use the same shading. Both cutoffs decrease with τ (Appendix C). As τ rises, the competing payoff V^C gains through both a higher price and a higher trading probability, whereas the local-monopoly payoff V^M gains only through the trading probability, so the pure-strategy region expands. For every τ , however, both mixed regimes remain nonempty.

4 Efficiency and price dispersion

The pure-strategy equilibrium of Case 1 (Proposition 2) coincides with that of BSW and is constrained efficient, since buyers visit the sellers in their sample with equal probability. Case 2, with its new pricing structure, has different efficiency implications, so this section focuses on the equilibrium constructed in Proposition 3.

Recall that $\varepsilon = \tau(1 - e^{-\mu})$ in Case 2. A seller's expected payoff $V(\mu, \tau)$ equals V^C in (9) if $\mu \geq \bar{\mu}$ and V^M in (10) if $\mu < \bar{\mu}$, and sellers' average trading probability is

$$\Psi^*(\mu, \tau) = \begin{cases} 1 - e^{-\varepsilon} & \text{if } \mu \geq \bar{\mu}(\tau) \\ \kappa \Psi(\hat{p}) + \int_{\underline{p}}^1 \Psi(p) dF^*(p) & \text{if } \mu < \bar{\mu}(\tau) \end{cases} \quad (20)$$

where the mixed-strategy branch has a closed form.⁷ Each trade generates one unit

⁷Specifically, when $\mu < \bar{\mu}$, $\Psi^*(\mu, \tau) = \kappa V^M / \hat{p} + [\text{Ein}(\varepsilon_{\underline{p}}) - \text{Ein}(\varepsilon_1)] / \mu$, where $\text{Ein}(x) \equiv \int_0^x (1 - e^{-t}) / t dt$ is the complementary exponential integral.

of surplus, so payoffs add up to the volume of trade per seller, $V + \tau U = \Psi^*$. A buyer thus expects $U = (\Psi^* - V)/\tau$, and the average payoff across all agents is $\Psi^*/(1 + \tau)$.

The first-best volume is the size of the short side, $\Psi^{(1)} = \min\{\tau, 1\}$. A constrained planner faces both frictions: it chooses how buyers visit the sellers in their samples but cannot coordinate buyers' visits. Because a seller's trading probability is concave in its queue and the mean queue is ε under any visiting rule, the planner has buyers visit sampled sellers with equal probability, which yields the constrained optimal volume $\Psi^{(2)} = 1 - e^{-\varepsilon}$.

Proposition 4. Fix τ . (i) Aggregate: Ψ^* is strictly increasing in μ , and $\Psi^* \leq \Psi^{(2)} < \Psi^{(1)}$, where the first inequality is strict iff $\mu < \bar{\mu}$. (ii) Buyers: If $\tau \leq 1$, U is strictly increasing in μ . If $\tau > 1$, U is single-peaked on $[\min\{1, \bar{\mu}\}, \infty)$, with its maximum at $\max\{\bar{\mu}, \mu^*\}$, where $\mu^*(\tau) \equiv -\ln(1 - 1/\tau)$. (iii) Sellers: V^M is maximized at $\mu = 1$. There is a unique $\hat{\tau} > 0$ (about 1.36) such that $\ln(1 + \hat{\tau}) = \hat{\tau}(1 - e^{-1})$; if $\tau \leq \hat{\tau}$, V is uniquely maximized at $\mu = 1$, and otherwise V approaches its supremum as $\mu \rightarrow \infty$.

Appendix E proves Proposition 4. By part (i), the equilibrium is constrained inefficient exactly in the mixed-strategy region: price dispersion spreads the queues ε_p around their mean ε , and because trade is concave in the queue, volume falls. The atom price satisfies a local Hosios condition, $1 - \hat{p} = \varepsilon_{\hat{p}} e^{-\varepsilon_{\hat{p}}}/(1 - e^{-\varepsilon_{\hat{p}}})$: buyers' share of the surplus equals the elasticity of the number of matches with respect to the number of buyers, evaluated at the atom's queue. In contrast, interval prices leave buyers a smaller share. This local condition does not deliver constrained efficiency, however, because atom sellers are themselves over-queued, $\varepsilon_{\hat{p}} > \varepsilon$.

Part (i) also shows that lower search friction always raises the average payoff $\Psi^*/(1 + \tau)$. It does so through a higher ε , which reflects fewer idle buyers, and, in the hybrid regime, through a larger atom, which improves matching. How this gain is split between buyers and sellers, however, changes with μ , and for every τ one side's payoff falls over some range of μ . Figure 6 plots U (blue), V (red), the average payoff (black dashed), and its first-best level $\Psi^{(1)}/(1 + \tau)$ (green dashed) against μ for $\tau \in \{0.5, 1, 2\}$.

For buyers, a higher μ has two opposing effects: it lowers the chance of being idle, but it intensifies competition among buyers, which raises p^* and makes each buyer more likely to be rationed.⁸ The balance depends on τ . In a buyers' market ($\tau \leq 1$),

⁸Lester (2011) shows that the price can either decrease or increase in the share of informed buyers because of buyer heterogeneity. Here, buyers are homogeneous and the pricing effect is unambiguous.

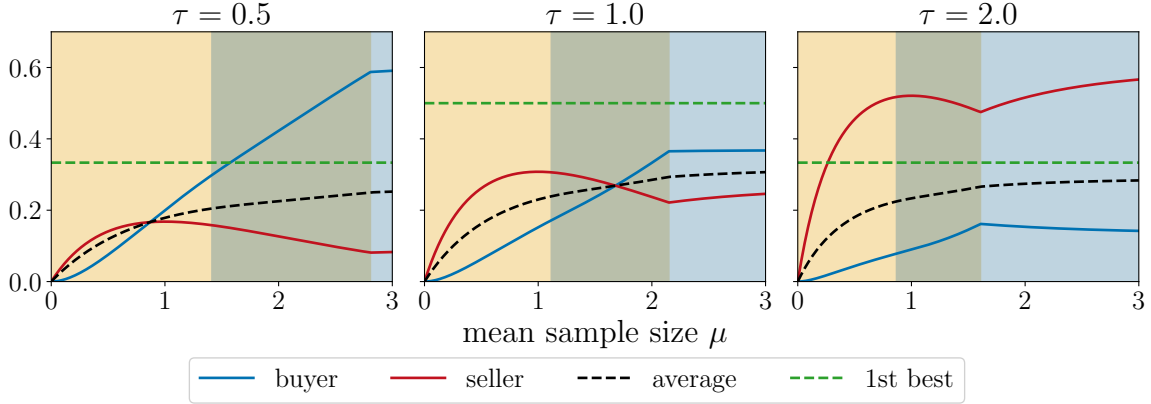


Figure 6: Case 2 payoffs and efficiency: $\tau \in \{0.5, 1, 2\}$.

the first effect dominates, and U increases monotonically. In a sellers' market ($\tau > 1$), the second effect dominates once μ exceeds $\max\{\bar{\mu}, \mu^*\}$, so U attains its maximum over $\mu \geq \min\{1, \bar{\mu}\}$ either at the cutoff $\bar{\mu}$ or at μ^* inside the pure-strategy region. Lower search friction can thus hurt buyers. For larger τ , U can also peak at a small μ inside the mixed region, which is why part (ii) starts at $\min\{1, \bar{\mu}\}$.

For sellers, the mixed-strategy payoff V^M comes from captive buyers alone and peaks at $\mu = 1$, where the rise in the number of buyers who sample a given seller, $\tau\mu$, just offsets the fall in the share of them who sample no rival, $e^{-\mu}$. Thus, when $\bar{\mu} > 1$, the seller's payoff has a local maximum in the mixed-strategy region. In the pure-strategy region, it increases monotonically in μ , as both the trading probability and the price rise. The local maximum is also the global one iff $\tau \leq \hat{\tau}$.

Buyers and sellers therefore never agree on the preferred level of μ . When $\tau \leq \hat{\tau}$, sellers prefer $\mu = 1$, whereas buyers' payoff is still rising at $\mu = 1$. When $\tau > \hat{\tau}$, sellers prefer the frictionless limit, whereas buyers strictly prefer some finite μ . Hence at least one side, and sometimes both, strictly prefers some $\mu < \infty$ to the frictionless limit, even though the average payoff increases in μ .

4.1 Efficiency decomposition

Let $L \equiv \Psi^{(1)} - \Psi^*$ be the total loss, the shortfall of equilibrium trade from the first best; in Figure 6, $L/(1 + \tau)$ is the gap between the green and black dashed lines. I decompose L into components associated with search and matching, using the counterfactual trade volumes summarized in Table 2.

Table 2: Efficiency decomposition

Panel A. Counterfactual volume under each combination of frictions.		
	Matching friction	
	absent	present
Search friction	(coordinated)	(urn-ball)
absent ($\mu \rightarrow \infty$)	$\Psi^{(1)} = \min\{\tau, 1\}$	$\Psi^{(\sigma)} = 1 - e^{-\tau}$
present ($\mu < \infty$)	$\Psi^{(\alpha)}$ (numerical)	$\Psi^{(2)} = 1 - e^{-\varepsilon}$
Equilibrium volume	Ψ^* by (20)	
Panel B. The four channels of efficiency loss.		
	Source	Loss
L_α isolation	search friction	$\Psi^{(1)} - \Psi^{(\alpha)}$
L_σ congestion	matching friction	$\Psi^{(1)} - \Psi^{(\sigma)}$
L_ε complementarity	search×matching	$\Psi^{(\alpha)} + \Psi^{(\sigma)} - \Psi^{(1)} - \Psi^{(2)}$
L_p distortion	price dispersion	$\Psi^{(2)} - \Psi^*$
Total loss $L = \Psi^{(1)} - \Psi^* = L_\alpha + L_\sigma + L_\varepsilon + L_p$. Effective tightness $\varepsilon = \tau(1 - e^{-\mu})$		

Panel A reports the trade volume under each of the four combinations of frictions: the first-best volume $\Psi^{(1)}$ with neither friction; the search-only volume $\Psi^{(\alpha)}$;⁹ the matching-only volume $\Psi^{(\sigma)} = \lim_{\mu \rightarrow \infty} \Psi^*(\mu, \tau) = 1 - e^{-\tau}$, equivalently $\Psi^{(2)}$ in the same limit; and the constrained optimum $\Psi^{(2)}$ with both frictions. Panel B decomposes L into four additive channels: isolation L_α , congestion L_σ , complementarity L_ε , and distortion L_p . The first two are the direct effects of search and matching, and the last two come from their interaction.

The direct effects are the gaps between the first best and the counterfactuals with a single friction: $L_\alpha = \Psi^{(1)} - \Psi^{(\alpha)}$ for search and $L_\sigma = \Psi^{(1)} - \Psi^{(\sigma)}$ for matching, where L_σ is also the entire loss in Case 1. The complementarity channel is a cross-difference.

⁹For the search-only counterfactual, $\Psi^{(\alpha)}$ is the size of a maximum matching, per seller, of the sampling graph, which links each buyer to the sellers it samples; in the large market, buyers' degrees are Poisson with mean μ and sellers' with mean $\tau\mu$. It satisfies $\Psi^{(2)} \leq \Psi^{(\alpha)} \leq \min\{\varepsilon, 1 - e^{-\tau\mu}\}$. The lower bound is the volume of the uncoordinated allocation on the same graph; the upper bound is the short side among active agents: buyers with a nonempty sample (ε per seller) and sellers sampled by at least one buyer (a share $1 - e^{-\tau\mu}$). I compute $\Psi^{(\alpha)}$ numerically. The upper bound already gives $L_\varepsilon < 0$ for $\tau \leq 1$: then $\Psi^{(1)} = \tau$, so $L_\varepsilon \leq (\varepsilon + e^{-\varepsilon}) - (\tau + e^{-\tau}) < 0$, as $x + e^{-x}$ increases in x and $\varepsilon < \tau$.

Adding search friction to a market that already has matching friction lowers trade by $\Psi^{(\sigma)} - \Psi^{(2)} = L_\alpha + L_\varepsilon$, so L_ε is the error committed by measuring one friction while holding the other fixed; symmetrically, $L_\sigma + L_\varepsilon = \Psi^{(\alpha)} - \Psi^{(2)}$.

The sign of L_ε indicates whether search and matching are substitutes or complements in destroying surplus. By construction, L_α , L_σ , and L_ε depend on the frictions alone and not on the price distribution: they are properties of the technology of exchange, shared by any pricing protocol with the same sampling and matching structure, while L_p is the only channel through which equilibrium behavior enters.

Finally, the distortion channel $L_p = \Psi^{(2)} - \Psi^*$ is the remaining gap between the constrained optimum and the equilibrium; its source is discussed after Proposition 4. Figure 7 reports the four channels as shares of L for $\tau \in \{0.2, 1, 5\}$. When μ is low, L_α is the major source of efficiency loss. As μ increases, L_σ overtakes L_α and eventually becomes the dominant source.

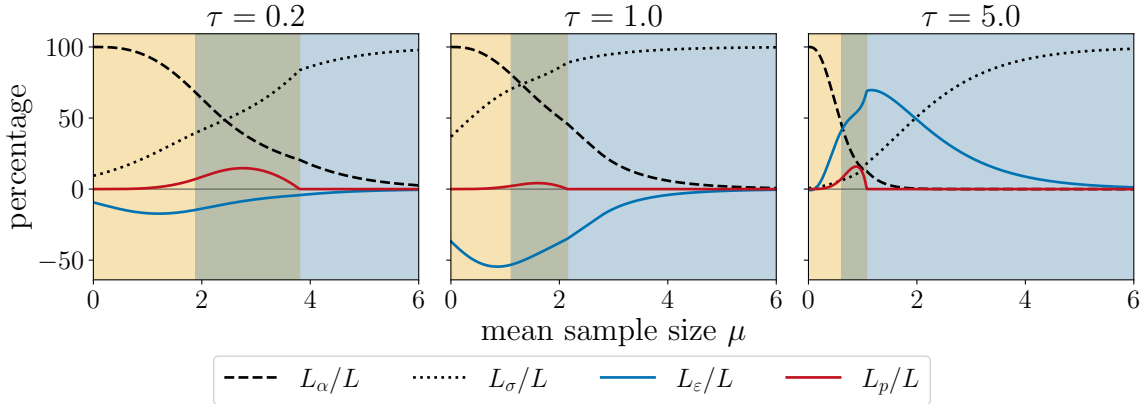


Figure 7: Efficiency decomposition: $\tau \in \{0.2, 1, 5\}$.

Being a cross-difference does not make the complementarity channel a mere residual: its sign depends on which side of the market is short. When $\tau \leq 1$, $L_\varepsilon \leq 0$: buyers are the short side, so an idle buyer costs a full trade under coordinated matching but less under urn-ball matching, where it might have been rationed anyway. Hence search and matching are substitutes in destroying surplus.¹⁰ When $\tau > 1$, L_ε can be positive: buyers are the long side, so an idle buyer is redundant under coordinated matching but still costs trades under urn-ball matching, where every active buyer adds matches. Hence the two frictions can be complements.

¹⁰As $\mu \rightarrow 0$, $L_\varepsilon \rightarrow -L_\sigma$: the complementarity channel exactly offsets the congestion channel, reflecting that all buyers are idle.

The distortion channel works through price dispersion, so $L_p > 0$ exactly in the mixed-strategy region. Price dispersion vanishes as $\mu \rightarrow 0$ and for $\mu \geq \bar{\mu}$, and L_p is hump-shaped in between. Quantitatively, L_ε can exceed half of L in absolute value, and L_p peaks at about 15% of L in the imbalanced markets ($\tau = 0.2$ and $\tau = 5$) but only about 5% in the balanced market ($\tau = 1$).

4.2 Price dispersion

Conventional wisdom holds that price dispersion signals efficiency loss from underlying frictions. With both search and matching frictions, price dispersion is an imperfect signal. Figure 8 plots the relative loss $L/\Psi^{(1)}$ and price dispersion, measured by the coefficient of variation (CV) of posted prices and of sales-weighted prices, for $\tau = 1$. As in noisy search, dispersion is hump-shaped in the mixed-strategy region, whereas the loss decreases monotonically in μ . Two implications follow.

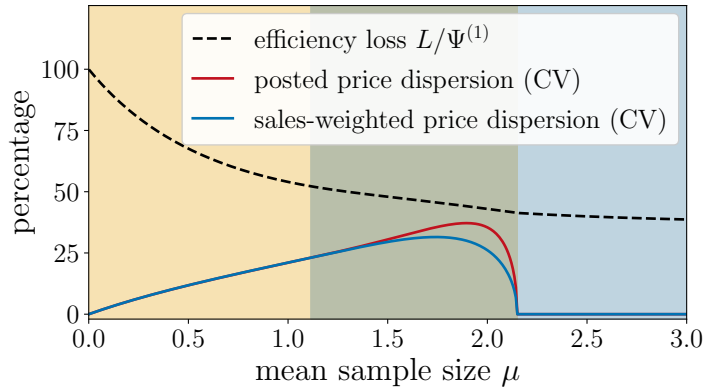


Figure 8: Efficiency loss vs. price dispersion: $\tau = 1$.

First, on the extensive margin, price dispersion is an exact marker of constrained inefficiency. It is sufficient but not necessary for first-best inefficiency, however, because the law of one price survives substantial search friction: price dispersion vanishes for $\mu \geq \bar{\mu}$, whereas $L > 0$ for every μ . Treating a market as efficient because its prices are not dispersed can therefore be misleading.

Second, on the intensive margin, price dispersion and efficiency loss are imperfectly correlated. Where dispersion decreases with μ , higher dispersion goes with a larger loss, consistent with conventional wisdom; where dispersion increases with μ , higher dispersion goes with a smaller loss. Noisy search can produce the same pattern, but

only because some buyers are idle. If every buyer sampled at least one seller, noisy search would achieve the first best at any level of dispersion, as every buyer would trade, whereas here dispersion would still destroy trade through rationing.

5 Conclusion

This paper models search and matching as distinct frictions in a price-posting game. The model is microfounded, admits closed-form equilibria in the large market, and generates price dispersion and rationing jointly. Search friction can distort the matching process, causing constrained inefficiency. This distortion operates entirely through price dispersion, so a policy that reshapes the price distribution affects both total surplus and its division. Reducing search friction raises total surplus but creates distributional conflict. Natural next steps include endogenizing search friction (Butters, 1977; Gomis-Porqueras, Julien, and Wang, 2017), allowing heterogeneous seller capacities (Watanabe, 2010), and evaluating price interventions in specific markets.

Appendices

A Proof of Lemma 1

The proof replaces the fixed-point problem in σ by the maximization of a potential function in θ , in the spirit of Monderer and Shapley (1996), whose maximizers are exactly the buyer equilibria.

Let Σ be the set of strategies, the product over nonempty samples S of the probability simplices on S ; it is nonempty, compact, and convex. By (1) the map $\sigma \mapsto \theta(\sigma)$ is linear, so the set of induced queue vectors $\Theta \equiv \theta(\Sigma)$ is also nonempty, compact, and convex; it depends on α and m but not on p . Define

$$\Omega_p(\theta) \equiv \sum_{j \in \mathcal{S}} (1 - p_j) \int_0^{\theta_j} \phi(s) ds. \quad (\text{A.1})$$

The partial derivative of Ω_p in θ_j is $(1 - p_j)\phi(\theta_j)$, the payoff from visiting seller j . Since ϕ is continuous and strictly decreasing, Ω_p is C^1 and concave, strictly concave

in each coordinate j with $p_j < 1$, and jointly continuous in $(\boldsymbol{\theta}, \mathbf{p})$.

Equivalence and existence. The claim is that $\boldsymbol{\sigma}^*$ is a buyer equilibrium iff $\boldsymbol{\theta}^* \equiv \boldsymbol{\theta}(\boldsymbol{\sigma}^*)$ maximizes $\Omega_{\mathbf{p}}$ over Θ , or equivalently, $\boldsymbol{\sigma}^*$ maximizes $\Omega_{\mathbf{p}} \circ \boldsymbol{\theta}$ over Σ . The latter function is concave and C^1 on the compact convex set Σ , so $\boldsymbol{\sigma}^*$ maximizes it iff $\nabla(\Omega_{\mathbf{p}} \circ \boldsymbol{\theta})(\boldsymbol{\sigma}^*) \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^*) \leq 0$ for every $\boldsymbol{\sigma} \in \Sigma$. Using $\partial\theta_j/\partial\sigma_j(\mathbf{p}_S) = \alpha^{|S|}(1 - \alpha)^{m-|S|}$ from (1), the condition reads

$$\sum_{S \neq \emptyset} \alpha^{|S|}(1 - \alpha)^{m-|S|} \sum_{j \in S} (1 - p_j) \phi(\theta_j^*) [\sigma_j(\mathbf{p}_S) - \sigma_j^*(\mathbf{p}_S)] \leq 0. \quad (\text{A.2})$$

Because Σ is a product over samples, (A.2) holds for every $\boldsymbol{\sigma} \in \Sigma$ iff, for every $S \neq \emptyset$, $\boldsymbol{\sigma}^*(\mathbf{p}_S)$ maximizes the linear function $\sum_{j \in S} (1 - p_j) \phi(\theta_j^*) \sigma_j(\mathbf{p}_S)$ over the simplex on S , that is, iff $\sigma_j^*(\mathbf{p}_S) > 0$ implies $j \in \arg \max_{k \in S} (1 - p_k) \phi(\theta_k^*)$. This is the best-response condition defining a buyer equilibrium. As the continuous $\Omega_{\mathbf{p}}$ attains its maximum on the compact Θ , a buyer equilibrium exists for every $\mathbf{p} \in [0, 1]^m$.

Uniqueness. If $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ both maximize $\Omega_{\mathbf{p}}$ over Θ and $\theta_j \neq \theta'_j$ for some j with $p_j < 1$, then their average, which lies in Θ , would be strictly better than both, as $\Omega_{\mathbf{p}}$ is additively separable, concave, and strictly concave in coordinate j . Hence θ_j^* is unique for every j with $p_j < 1$, and if at most one seller posts 1, the adding-up identity $\sum_j \theta_j = 1 - (1 - \alpha)^m$ pins down the remaining coordinate.

Continuity. Since Θ does not depend on $\mathbf{p}$ and $\Omega_{\mathbf{p}}(\boldsymbol{\theta})$ is jointly continuous, Berge's maximum theorem makes the set of maximizers, and each of its coordinates, upper hemicontinuous in $\mathbf{p}$. An upper hemicontinuous correspondence is continuous wherever it is single-valued, so θ_j^* is continuous in $\mathbf{p}$ wherever $p_j < 1$, and $\boldsymbol{\theta}^*$ wherever at most one seller posts 1. ■

B Proof of Proposition 1

Throughout, $\alpha < \bar{\alpha}$, so that $V^C(\alpha) < V^M(\alpha)$. Note also that $\underline{\theta} < \theta^* < \alpha$, since $m\theta^* = 1 - (1 - \alpha)^m = \alpha \sum_{k=0}^{m-1} (1 - \alpha)^k$. First come (i) and (ii), which concern the top of the support and imply that every equilibrium is mixed, then (iii), which concerns its bottom. Both steps rest on how $\boldsymbol{\theta}^*$ responds to $\mathbf{p}$. Let $u_j^* \equiv (1 - p_j) \phi(\theta_j^*)$ denote the payoff from visiting seller j in a buyer equilibrium.

Ranking and blocks. The first structural fact is that buyers rank sellers by price.

Lemma B.1. In any buyer equilibrium, $p_j \leq p_k$ implies $u_j^* \geq u_k^*$.

Proof. First, $u_j^* > u_k^*$ implies $\theta_j^* \geq \theta_k^*$. Let S be any sample with $\sigma_k^*(\mathbf{p}_S) > 0$, so that k attains $\max_{l \in S} u_l^*$ and hence $j \notin S$. In $S' \equiv (S \setminus \{k\}) \cup \{j\}$ seller j is the unique maximizer, so $\sigma_j^*(\mathbf{p}_{S'}) = 1 \geq \sigma_k^*(\mathbf{p}_S)$. The map $S \mapsto S'$ is injective and preserves $|S|$, hence the sampling probability, so summing (1) over these samples gives $\theta_j^* \geq \theta_k^*$. Now suppose $p_j \leq p_k$ but $u_j^* < u_k^*$. Then $\theta_k^* \geq \theta_j^*$, so $\phi(\theta_j^*) \geq \phi(\theta_k^*)$ and $u_j^* = (1 - p_j)\phi(\theta_j^*) \geq (1 - p_k)\phi(\theta_k^*) = u_k^*$, a contradiction. ■

Lemma B.1 reduces the analysis from global to local competition. Relabel sellers so that $p_1 \leq \dots \leq p_m$, hence $u_1^* \geq \dots \geq u_m^*$, and group consecutive sellers with a common value of u^* . This partitions $\mathcal{S}$ into at most m blocks, ordered by strictly decreasing payoff. Sellers posting the same price always share a block, and a seller posting $p_k = 1$ never shares a block with one posting $p_j < 1$, since $u_k^* = 0 < u_j^*$. Call the number of sellers in strictly better blocks the *level* of a block, and write $\mathcal{B}_j$ and M_j for the block and the level of seller j . A buyer visits a member of a block $\mathcal{B}$ at level M exactly when it samples some seller in $\mathcal{B}$ and none in a better block, so the mass reaching a block is fixed by the partition alone:

$$\sum_{j \in \mathcal{B}} \theta_j^* = (1 - \alpha)^M [1 - (1 - \alpha)^{|\mathcal{B}|}]. \quad (\text{B.1})$$

Individual queues are bounded in the same way.

Lemma B.2. In any buyer equilibrium,

$$\alpha(1 - \alpha)^{M_j + |\mathcal{B}_j| - 1} \leq \theta_j^* \leq \alpha(1 - \alpha)^{M_j}, \quad (\text{B.2})$$

with equality on the right when j is alone in its block.

Proof. A buyer who samples j but no seller in a better block or in $\mathcal{B}_j \setminus \{j\}$ visits j with certainty, which gives the lower bound. A buyer visits j only if it samples j and no seller in a better block, an event of probability $\alpha(1 - \alpha)^{M_j}$, which gives the upper bound; when j is alone in its block, every such buyer visits j . ■

Curvature. For a fixed $\mathbf{p}_{-j}$, seller j 's queue θ_j^* and payoff $\tilde{V}(p_j; \mathbf{p}_{-j}) \equiv p_j \psi(\theta_j^*)$ are continuous in p_j on $[0, 1)$ by Lemma 1. The curvature of θ_j^* in p_j rests on two properties of ϕ .

Lemma B.3. The functions ϕ and $1/\phi$ are convex on $[0, 1]$.

Proof. Since $\phi(\theta) = \frac{1}{n} \sum_{k=0}^{n-1} (1 - \theta)^k$ is a sum of convex functions, $\phi'' \geq 0$. For $1/\phi = n\theta/\psi(\theta)$, differentiating twice gives

$$\left(\frac{\theta}{\psi}\right)'' = \frac{-\theta\psi\psi'' - 2\psi'(\psi - \theta\psi')}{\psi^3}.$$

Substituting $x \equiv 1 - \theta$, $\psi = 1 - x^n$, $\psi' = nx^{n-1}$, $\psi'' = -n(n-1)x^{n-2}$ and $1 - x^n = (1 - x) \sum_{k=0}^{n-1} x^k$, the numerator equals

$$nx^{n-2}(1-x) \left[(n-1)(1+x^n) - 2 \sum_{k=1}^{n-1} x^k \right] = nx^{n-2}(1-x) \sum_{k=1}^{n-1} (1-x^k)(1-x^{n-k}),$$

which is nonnegative on $[0, 1]$. ■

Let $\varrho(\theta) \equiv -\phi(\theta)/\phi'(\theta) = \theta/[1 - \eta(\theta)]$, where the second form holds because $\psi = n\theta\phi$ implies $\theta\phi'(\theta)/\phi(\theta) = \eta(\theta) - 1$. In a buyer equilibrium, the *effective block size* of a seller j with $p_j < 1$ is

$$\beta_j \equiv \frac{\sum_{k \in \mathcal{B}_j} \varrho(\theta_k^*)}{\varrho(\theta_j^*)} \geq 1,$$

which equals 1 when j is alone in its block and $|\mathcal{B}_j|$ when all members of $\mathcal{B}_j$ post the same price.

Lemma B.4. Fix $\mathbf{p}_{-j}$ and suppose $p_j < 1$ and j shares its block $\mathcal{B}$ with at least one other seller. Then, on any interval of p_j over which the partition is unchanged, θ_j^* and u_j^* are strictly decreasing in p_j , θ_j^* is concave, and

$$\frac{d\theta_j^*}{dp_j} = -\frac{\varrho(\theta_j^*)}{1 - p_j} \left[1 - \frac{1}{\beta_j} \right] < 0. \quad (\text{B.3})$$

Proof. Let u be $\mathcal{B}$'s common payoff and Q its mass (B.1), fixed by the partition. Every $k \in \mathcal{B}$ has $p_k < 1$ and $\theta_k^* = \phi^{-1}(u/(1 - p_k))$, so u and θ_j^* are tied together by

$$G(u) = Q - \theta_j^*, \quad G(u) \equiv \sum_{k \in \mathcal{B} \setminus \{j\}} \phi^{-1}(u/(1 - p_k)).$$

Since ϕ is decreasing and convex (Lemma B.3), so is ϕ^{-1} , hence so is G , and hence so is G^{-1} . Therefore the block payoff as a function of j 's queue, $\tilde{u}(\theta_j) \equiv G^{-1}(Q - \theta_j)$, is increasing and convex. Now $1 - p_j = \tilde{u}(\theta_j^*)/\phi(\theta_j^*)$, and the right side is a product of two positive, increasing, convex functions of θ_j^* , hence increasing and convex, and strictly increasing as $1/\phi$ is. Inverting, θ_j^* is strictly decreasing and concave in p_j , and $u_j^* = \tilde{u}(\theta_j^*)$ is strictly decreasing. For (B.3), differentiate $(1 - p_k)\phi(\theta_k^*) = u$ for each $k \in \mathcal{B}$ and $\sum_{k \in \mathcal{B}} \theta_k^* = Q$ with respect to p_j , as the implicit function theorem allows, writing dots for these derivatives. Dividing the former by $u = (1 - p_k)\phi(\theta_k^*)$ gives $\dot{\theta}_k = -\varrho(\theta_k^*) \dot{u}/u$ for $k \neq j$ and $\dot{\theta}_j = -\varrho(\theta_j^*) [\dot{u}/u + 1/(1 - p_j)]$. Then $\sum_{k \in \mathcal{B}} \dot{\theta}_k = 0$ gives $\dot{u}/u = -1/[(1 - p_j)\beta_j]$, and substituting yields (B.3). ■

Kinks. Fix $\mathbf{p}_{-j}$ and raise p_j from 0 to 1. While the partition is unchanged, u_j^* falls strictly: by Lemma B.4 when j shares a block, and because θ_j^* is constant at $\alpha(1 - \alpha)^{M_j}$ by Lemma B.2 when j is alone. Every other block's payoff is determined by its own prices and its mass (B.1), so it is constant while the partition is unchanged. Blocks above j therefore never meet the falling u_j^* , blocks not containing j never meet each other, and the partition changes only at *junctions* of two kinds. At a *join*, u_j^* falls to the payoff of the block just below and the two merge: M_j is unchanged and $|\mathcal{B}_j|$ rises. At a *split*, j 's block breaks into a strictly better part $\mathcal{B}^-$ and the rest $\mathcal{B}^+$, and $j \in \mathcal{B}^+$: otherwise $\mathcal{B}^+$'s payoff would be constant after the split and, by continuity, equal to the common payoff at the split, while u_j^* keeps falling below it. So M_j rises by $|\mathcal{B}^-|$. Since M_j only rises and the number of sellers in worse blocks than j only falls, there are at most $m - 1$ junctions of each kind.

At a junction, θ_j^* and the queues in j 's block are continuous by Lemma 1, so only β_j jumps in (B.3), which also holds when j is alone, with $\beta_j = 1$ and a zero derivative. A join adds sellers to $\mathcal{B}_j$ and raises β_j , and a split removes $\mathcal{B}^-$ and lowers it, so $d\theta_j^*/dp_j$ jumps strictly down at a join and strictly up at a split: the more sellers share j 's block, the more buyers a price increase diverts. Thus θ_j^* has a *concave kink* at each join and a *convex kink* at each split, and between junctions it is constant when j is alone and concave by Lemma B.4 otherwise. Hence θ_j^* is concave in p_j on any interval whose interior contains no split, and so is $\psi(\theta_j^*)$, as ψ is increasing and concave. The next lemma separates sellers in different blocks, and so places the splits.

Lemma B.5. Let $\gamma \equiv 1 - \alpha^2(1 - \alpha)^{m-2}/n \in (0, 1)$ and $w(p) \equiv 1 - \gamma(1 - p)$, an increasing function with $w(p) > p$ for $p < 1$. In any buyer equilibrium, if k is in a

strictly better block than j , then $p_j > w(p_k)$. Consequently, (a) if $p_k \leq p_j \leq w(p_k)$, then j and k share a block; and (b) if j 's block splits at $p_j = P < 1$, leaving k in the better part, then $P \geq w(p_k)$.

Proof. As k is in a strictly better block, $M_k + |\mathcal{B}_k| \leq M_j \leq m - 1$, so (B.2) gives $\theta_j^* \leq \alpha(1 - \alpha)^{M_k + |\mathcal{B}_k|} \leq (1 - \alpha)\theta_k^*$ and $\theta_k^* \geq \alpha(1 - \alpha)^{m-2}$. Hence $\theta_k^* - \theta_j^* \geq \alpha\theta_k^* \geq \alpha^2(1 - \alpha)^{m-2}$. Since $-\phi'(\theta) = \frac{1}{n} \sum_{i=1}^{n-1} i(1 - \theta)^{i-1} \geq 1/n$ and $\phi \leq 1$,

$$\frac{\phi(\theta_k^*)}{\phi(\theta_j^*)} \leq 1 - [\phi(\theta_j^*) - \phi(\theta_k^*)] \leq 1 - \frac{\alpha^2(1 - \alpha)^{m-2}}{n} = \gamma,$$

so $(1 - p_k)\phi(\theta_k^*) = u_k^* > u_j^* = (1 - p_j)\phi(\theta_j^*) \geq (1 - p_j)\phi(\theta_k^*)/\gamma$, that is, $1 - p_j < \gamma(1 - p_k)$, or $p_j > w(p_k)$. For (a), k is then not in a strictly better block than j , while $u_k^* \geq u_j^*$ by Lemma B.1. For (b), k is in a strictly better block than j for p_j slightly above P , and letting $p_j \downarrow P$ gives $P \geq w(p_k)$. $\blacksquare$

The top of the support. Fix an equilibrium F^* , and let $V^* \equiv \max_p V(p; F^*)$ and $\bar{c} \equiv \max \text{supp}(F^*)$, which exists because the support is closed and nonempty. The first step concerns the monopoly price itself.

Lemma B.6. In any equilibrium, $F^*(\{1\}) = 0$ and $V(1; F^*) = V^M(\alpha)$.

Proof. Suppose $F^*(\{1\}) > 0$, so that $1 \in \text{supp}(F^*)$. Queues need not be unique when several sellers post 1 (Lemma 1), but whichever buyer equilibrium is selected there, the queues sum to $m\theta^*$ when all sellers post 1, so some seller j then has a queue of at most θ^* . Fix $\mathbf{p}_{-j}$ and let M be the number of rivals posting less than 1. At $p_j = 1$, these rivals occupy the better blocks, so j is at level M and $\theta_j^* \leq \alpha(1 - \alpha)^M$ by Lemma B.2. At any $p_j < 1$, every rival posting 1 is in a strictly worse block than j , as $u_k^* = 0 < u_j^*$, so $M_j + |\mathcal{B}_j| - 1 \leq M$ and $\theta_j^* \geq \alpha(1 - \alpha)^M$ by Lemma B.2. Hence, for every $\delta \in (0, 1)$,

$$\begin{aligned} V(1 - \delta; F^*) &\geq (1 - \delta) \mathbb{E}[\psi(\alpha(1 - \alpha)^M)] \\ &\geq (1 - \delta) \{V(1; F^*) + F^*(\{1\})^{m-1}[\psi(\alpha) - \psi^*]\}, \end{aligned}$$

where the last term comes from the profiles in which every rival posts 1: there $M = 0$ and $\theta_j^* \leq \theta^* < \alpha$. For small δ the right side exceeds $V(1; F^*)$, which contradicts the optimality of $1 \in \text{supp}(F^*)$ for j . So $F^*(\{1\}) = 0$, and a seller posting 1 almost surely

faces rivals that all post less than 1; it is then alone in the last block, at level $m - 1$, and earns $\psi(\underline{\theta}) = V^M(\alpha)$. $\blacksquare$

Next consider a seller j with $p_j < 1$. By (B.3) and $\theta\psi'(\theta) = \eta(\theta)\psi(\theta)$, on any interval of p_j over which the partition is unchanged,

$$\tilde{V}'(p_j; \mathbf{p}_{-j}) = \frac{\psi(\theta_j^*)}{1 - p_j} \left(1 - \frac{p_j}{\pi_j}\right), \quad \pi_j \equiv \frac{\beta_j [1 - \eta(\theta_j^*)]}{\beta_j - \eta(\theta_j^*)} \in (0, 1]. \quad (\text{B.4})$$

So π_j is the price at which j 's marginal payoff would vanish if its queue and its block stayed as they are. When all m sellers post the same price, $\beta_j = m$ and $\pi_j = p^*$ by (5), so $\pi_j\psi(\theta_j^*) = V^C$. Writing $\pi_j\psi(\theta_j^*) = \beta_j[1 - \eta(\theta_j^*)]\psi(\theta_j^*)/[\beta_j - \eta(\theta_j^*)]$ shows that, whenever $[1 - \eta(\theta_j^*)]\psi(\theta_j^*) < V^M$,

$$\pi_j\psi(\theta_j^*) \leq V^M \iff \beta_j \geq \beta^\dagger(\theta_j^*), \quad \beta^\dagger(\theta) \equiv \frac{V^M\eta(\theta)}{V^M - [1 - \eta(\theta)]\psi(\theta)}. \quad (\text{B.5})$$

Lemma B.7. On $[\underline{\theta}, \theta^*]$, $[1 - \eta(\theta)]\psi(\theta) < V^M$, and $\beta^\dagger$ is convex, with $\beta^\dagger(\underline{\theta}) = 1$ and $\beta^\dagger(\theta^*) < m$.

Proof. The function $(1 - \eta)\psi = \psi - \theta\psi'$ has derivative $-\theta\psi'' > 0$, and at θ^* it is below $p^*\psi^* = V^C < V^M$, as $p^* > 1 - \eta^*$ by (5); this proves the first claim. Next, $V^M/\beta^\dagger = \psi - (\psi - V^M)/\eta$. In $x \equiv 1 - \theta$ and $\bar{x} \equiv 1 - \underline{\theta}$, $\psi = 1 - x^n$, $V^M = 1 - \bar{x}^n$ and $1/\eta = \frac{1}{n} \sum_{i=0}^{n-1} x^{-i}$, so

$$\frac{V^M}{\beta^\dagger} = 1 - x^n - \frac{1}{n} \left[\bar{x}^n \sum_{i=0}^{n-1} x^{-i} - \sum_{i=1}^n x^i \right].$$

Since $\bar{x}^n x^{-i-2} \geq x^{n-i-2}$ for $0 < x \leq \bar{x}$, that is, for $\theta \geq \underline{\theta}$, substituting $i \rightarrow n - i$ in the first sum gives

$$\begin{aligned} -n \frac{d^2}{dx^2} \left(\frac{V^M}{\beta^\dagger} \right) &\geq \sum_{i=1}^n [(n-i)(n-i+1) - i(i-1)] x^{i-2} + n^2(n-1)x^{n-2} \\ &= n \sum_{i=1}^n (n+1-2i) x^{i-2} + n^2(n-1)x^{n-2}. \end{aligned}$$

Pairing the terms i and $n+1-i$ turns the sum in the last line into $\sum_{2i < n+1} (n+1-2i)(x^{i-2} - x^{n-1-i}) \geq 0$, as $x < 1$. Hence $V^M/\beta^\dagger$ is concave in x , and so in θ ; by the

first claim it is positive on $[\underline{\theta}, \theta^*]$, so $\beta^\dagger$ is convex there. At $\underline{\theta}$, $V^M - [1 - \eta]\psi = \underline{\theta}\psi'(\underline{\theta}) = V^M\eta(\underline{\theta})$, so $\beta^\dagger(\underline{\theta}) = 1$; and $\beta^\dagger(\theta^*) < m$ reduces to $m(1 - \eta^*)\psi^* < (m - \eta^*)V^M$, that is, to $V^C < V^M$. $\blacksquare$

Lemma B.8. Let $\omega_n \equiv \max\{0, (5 - n)/[3(n - 1)]\}$. Then $\varrho \geq 2/[n(n - 1)] \geq \omega_n$ and $\varrho' \geq -\omega_n$ on $[0, 1]$.

Proof. In $x \equiv 1 - \theta$, $\phi = \frac{1}{n} \sum_{i=0}^{n-1} x^i$, so $\varrho = \phi/\phi_x$ and $\varrho'(\theta) = (\phi\phi_{xx} - \phi_x^2)/\phi_x^2$, where subscripts denote derivatives in x . First, $\phi \geq 1/n$ and $\phi_x \leq (n - 1)/2$ give $\varrho \geq 2/[n(n - 1)]$, which is at least ω_n for every n . Second, for $n \leq 4$ direct computation gives $\varrho = 2 - \theta$ if $n = 2$, and

$$\begin{aligned} \varrho' + \frac{1}{3} &= \frac{2(1 - x)(2 + x)}{3(1 + 2x)^2} && \text{if } n = 3, \\ \varrho' + \frac{1}{9} &= \frac{(1 - x)(18x^3 + 42x^2 + 50x + 10)}{9(1 + 2x + 3x^2)^2} && \text{if } n = 4, \end{aligned}$$

and both right sides are nonnegative on $[0, 1]$. For $n \geq 5$, expanding and symmetrizing the double sums,

$$2n^2(\phi\phi_{xx} - \phi_x^2) = \sum_{i,l=0}^{n-1} [(i - l)^2 - (i + l)]x^{i+l-2} = \sum_{k=0}^{2n-2} a_k x^{k-2},$$

where $a_k = (k - 1)k(k + 1)/3 \geq 0$ for $k \leq n - 1$, and $a_k = (K + 1)[K(K + 2)/3 - k]$ with $K \equiv 2n - 2 - k$ for $k \geq n$, where the bracket decreases in k . So the a_k are nonnegative up to some k and negative after it, and their partial sums are at least $\min\{a_0, \sum_k a_k\} = \min\{0, n^2(n - 1)(n - 5)/6\} = 0$, the total being the value of the sum at $x = 1$. As x^{k-2} decreases in k , summation by parts gives $\phi\phi_{xx} \geq \phi_x^2$, that is, $\varrho' \geq 0$. $\blacksquare$

Lemma B.9. In any buyer equilibrium, let j post the highest price, $p_j < 1$, in the last block $\mathcal{B}$, and suppose $s \equiv |\mathcal{B}| \geq 2$. Then $\beta_j > \beta^\dagger(\theta_j^*)$, and so $\pi_j\psi(\theta_j^*) < V^M$.

Proof. The last block sits at level $m - s$, so by (B.1) the average queue of its members is $\vartheta_s \equiv (1 - \alpha)^{m-s}[1 - (1 - \alpha)^s]/s$, and every member has a queue of at least $\underline{\theta}$ by (B.2). Expanding $(1 - \alpha)^{-s} = \exp[-s \ln(1 - \alpha)]$, $\vartheta_s = (1 - \alpha)^m[(1 - \alpha)^{-s} - 1]/s = (1 - \alpha)^m \sum_{i \geq 1} [-\ln(1 - \alpha)]^i s^{i-1}/i!$ is convex and increasing in real $s \in [1, m]$, with $\vartheta_1 = \underline{\theta}$ and $\vartheta_m = \theta^*$, so $\vartheta_s \leq \theta^*$ and $(\vartheta_s - \underline{\theta})/(s - 1) \leq (\theta^* - \underline{\theta})/(m - 1)$. As

$\phi(\theta_k^*) = u/(1 - p_k)$ for every $k \in \mathcal{B}$, where u is the common payoff of $\mathcal{B}$, the most expensive member has the smallest queue, so $t \equiv \theta_j^* \in [\underline{\theta}, \vartheta_s] \subseteq [\underline{\theta}, \theta^*]$, and (B.5) applies by Lemma B.7. If $t = \underline{\theta}$, then $\beta_j > 1 = \beta^\dagger(\underline{\theta})$, as j has block-mates. Otherwise, as $\beta^\dagger$ is convex with $\beta^\dagger(\underline{\theta}) = 1$ and $\beta^\dagger(\theta^*) < m$, it lies below the line through $(\underline{\theta}, 1)$ and (θ^*, m) :

$$\beta^\dagger(t) < 1 + (m - 1) \frac{t - \underline{\theta}}{\theta^* - \underline{\theta}}.$$

On the other hand, every block-mate has a queue $\theta_k^* \geq t$, and $\sum_{k \in \mathcal{B} \setminus \{j\}} (\theta_k^* - t) = s(\vartheta_s - t)$, so Lemma B.8 gives $\sum_{k \in \mathcal{B} \setminus \{j\}} \varrho(\theta_k^*) \geq (s - 1)\varrho(t) - \omega_n s(\vartheta_s - t)$, and hence

$$\beta_j - 1 \geq (s - 1) - \frac{\omega_n s(\vartheta_s - t)}{\varrho(t)} \geq (s - 1) \frac{t - \underline{\theta}}{\vartheta_s - \underline{\theta}} \geq (m - 1) \frac{t - \underline{\theta}}{\theta^* - \underline{\theta}}.$$

The second inequality reduces to $(s - 1)\varrho(t) \geq \omega_n s(\vartheta_s - \underline{\theta})$, which holds because $(s - 1)\varrho(t) \geq (s - 1)\omega_n \geq s\omega_n/2$ and $\vartheta_s - \underline{\theta} < \theta^* < 1/m \leq 1/2$; the third is the slope comparison above. Combining the two displays gives $\beta_j > \beta^\dagger(t)$, and (B.5) gives $\pi_j \psi(\theta_j^*) < V^M$. $\blacksquare$

Proof of (i) and (ii). Suppose that $\bar{c} < 1$. Every queue is at least $\underline{\theta}$ by (B.2), so $V^* \geq V(1; F^*) \geq V^M > 0 = V(0; F^*)$, while $V(\bar{c}; F^*) = V^*$ as $\bar{c} \in \text{supp}(F^*)$; hence $\bar{c} \in (0, 1)$ maximizes $V(\cdot; F^*)$. Fix $\mathbf{p}_{-j} \in \text{supp}(F^*)^{m-1}$, so that every rival posts at most $\bar{c}$, and let p_j vary in $(w^{-1}(\bar{c}), w(\bar{c})) \cap [0, 1)$, a neighborhood of $\bar{c}$, where $w^{-1}(\bar{c}) = 1 - (1 - \bar{c})/\gamma$. Then j is in the last block: if $p_j \leq \bar{c}$, every rival priced in $[p_j, \bar{c}]$ shares j 's block by Lemma B.5(a), as $\bar{c} < w(p_j)$; if $p_j > \bar{c}$, j is the most expensive seller. So as p_j rises over this interval no join occurs, and $\tilde{V}' = \psi(\theta_j^*) + p_j \psi'(\theta_j^*) d\theta_j^*/dp_j$ jumps strictly up at each of the finitely many splits, as $d\theta_j^*/dp_j$ does. Between splits, (B.3) gives $|\tilde{V}'| \leq \psi(\theta_j^*) + p_j \psi'(\theta_j^*) \varrho(\theta_j^*)/(1 - p_j) \leq 1 + n^2/[\gamma(1 - \bar{c})]$, as $\psi' \leq n$, $\varrho \leq n$ and $1 - p_j > 1 - w(\bar{c}) = \gamma(1 - \bar{c})$, so the difference quotients of $\tilde{V}(\cdot; \mathbf{p}_{-j})$ over this interval are bounded uniformly in $\mathbf{p}_{-j}$. By dominated convergence, $V(\cdot; F^*)$ has one-sided derivatives at $\bar{c}$, and

$$V'_+(\bar{c}; F^*) - V'_-(\bar{c}; F^*) = \mathbb{E}[\tilde{V}'_+(\bar{c}; \mathbf{p}_{-j}) - \tilde{V}'_-(\bar{c}; \mathbf{p}_{-j})],$$

the expectation of a nonnegative variable that is positive exactly when j 's block splits at $\bar{c}$. Since $\bar{c}$ maximizes V , the left side is nonpositive, so almost surely no

split occurs at $\bar{c}$. In every other profile the partition is constant near $\bar{c}$, and (B.4) gives $\tilde{V}'(\bar{c}; \mathbf{p}_{-j}) = \psi(\theta_j^*) (1 - \bar{c}/\pi_j)/(1 - \bar{c})$, with θ_j^* and π_j evaluated at $p_j = \bar{c}$ here and below; if j is alone, $\pi_j = 1$ and $\psi(\theta_j^*) = \psi(\underline{\theta}) = V^M$. Taking expectations, $V'(\bar{c}; F^*) = 0$ gives $\mathbb{E}[\psi(\theta_j^*)] = \bar{c} \mathbb{E}[\psi(\theta_j^*)/\pi_j]$, so by the Cauchy–Schwarz inequality

$$V^* = \bar{c} \mathbb{E}[\psi(\theta_j^*)] = \frac{\mathbb{E}[\psi(\theta_j^*)]^2}{\mathbb{E}[\psi(\theta_j^*)/\pi_j]} \leq \mathbb{E}[\pi_j \psi(\theta_j^*)].$$

At $p_j = \bar{c}$, j posts the highest price in the last block, so $\pi_j \psi(\theta_j^*) \leq V^M$ in every profile, with strict inequality by Lemma B.9 whenever j has a block-mate. That happens with probability at least $1 - [1 - F^*((w^{-1}(\bar{c}), \bar{c}])]^{m-1} > 0$, since a rival priced in $(w^{-1}(\bar{c}), \bar{c}]$ shares j 's block by Lemma B.5(a). Hence $V^* < V^M$, a contradiction, so $\bar{c} = 1$.

As $1 \in \text{supp}(F^*)$ and $F^*({1}) = 0$ by Lemma B.6, the point 1 is not isolated in the support, so $F^*((q, 1)) > 0$ for every $q < 1$, which is (i). Finally, $V^* = V(1; F^*) = V^M(\alpha)$ by Lemma B.6, which is (ii).

The bottom of the support. Let $\underline{c} \equiv \min \text{supp}(F^*)$; by (i), $\underline{c} < 1$, and $\underline{c} > 0$ as $V(0; F^*) = 0 < V^*$. Every coordinate of $\mathbf{p}_{-j} \in \text{supp}(F^*)^{m-1}$ is at least $\underline{c}$, so by Lemma B.5(b), as w is increasing, no split occurs below $w(\underline{c}) < 1$, and $\psi(\theta_j^*)$ is concave in p_j on $[\underline{c}, w(\underline{c})]$. Hence so is its expectation $V(p_j; F^*)/p_j$. If this interval contained another point q of $\text{supp}(F^*)$, then $V(p; F^*)/p$ would equal V^*/p at $p = \underline{c}$ and at $p = q$ and, being concave, would exceed the strictly convex V^*/p between them, so that $V(p; F^*) > V^*$, which is impossible. So the interval meets $\text{supp}(F^*)$ only at $\underline{c}$. Being isolated in the support, $\underline{c}$ carries positive mass, and $(\underline{c}, w(\underline{c}))$, an interval of length $(1 - \gamma)(1 - \underline{c}) > 0$, contains no point of the support. Thus, the lowest price in $\text{supp}(F^*)$ is an atom followed by a gap, which is (iii). ■

C Details for the cutoff rules

This appendix first establishes the finite-market cutoff $\bar{\alpha}$ of Section 2 and its comparative statics in n , then the large-market cutoffs $\bar{\mu}$ and $\hat{\mu}$ of Section 3, and finally, using the large-market functions, the comparative statics of $\bar{\alpha}$ in m .

The finite-market cutoff $\bar{\alpha}$. Recall that $V^C(\alpha) = p^* \psi^*$ and $V^M(\alpha) = \psi(\underline{\theta})$. A single crossing is not immediate from the payoffs themselves, because V^M is not monotone:

it is single-peaked at $\alpha = 1/m$, since $\underline{\theta}'(\alpha) = (1 - \alpha)^{m-2}(1 - m\alpha)$. Instead, write

$$\Upsilon \equiv \frac{\psi^* - \psi(\underline{\theta})}{\theta^* \psi'(\theta^*)} \quad (\text{C.1})$$

for the gain in trading probability from competing rather than serving only captive buyers, in units of the marginal gain $\theta^* \psi'(\theta^*) = \eta^* \psi^*$. Since $1 - p^* = (m - 1)\eta^*/(m - \eta^*)$ by (5), $V^C \gtrless V^M$ exactly as $\psi^* - \psi(\underline{\theta}) \gtrless (1 - p^*) \psi^*$, and dividing by $\theta^* \psi'(\theta^*)$,

$$V^C \gtrless V^M \iff \Upsilon \gtrless \frac{m - 1}{m - \eta^*}. \quad (\text{C.2})$$

Lemma C.2 shows that Υ rises with α while $(m - 1)/(m - \eta^*)$ falls. The monotonicity rests on one observation about the family

$$\psi_\nu(\theta) \equiv 1 - (1 - \theta)^\nu, \quad \eta_\nu(\theta) \equiv \frac{\theta \psi'_\nu(\theta)}{\psi_\nu(\theta)}, \quad \nu \geq 2, \theta \in (0, 1), \quad (\text{C.3})$$

of which $\psi = \psi_n$ and $\eta = \eta_n$ are the members with $\nu = n$, while the two queues come from the member with $\nu = m$: $\theta^* = \psi_m(\alpha)/m$ and $\underline{\theta} = \alpha \psi'_m(\alpha)/m$.

Lemma C.1. For every integer $\nu \geq 2$ and $\theta \in (0, 1)$,

$$\frac{1}{\eta_\nu(\theta)} = \frac{1}{\nu} \sum_{k=0}^{\nu-1} (1 - \theta)^{-k}, \quad (\text{C.4})$$

so $\eta_\nu(\theta) \in (0, 1)$ is strictly decreasing in θ and in ν . In particular, the captive share of the queue $\chi \equiv \underline{\theta}/\theta^* = \eta_m(\alpha)$ is strictly decreasing in α and in m .

Proof. Since $\psi'_\nu(\theta) = \nu(1 - \theta)^{\nu-1}$ and $\psi_\nu(\theta) = \theta \sum_{k=0}^{\nu-1} (1 - \theta)^k$,

$$\frac{1}{\eta_\nu(\theta)} = \frac{\sum_{k=0}^{\nu-1} (1 - \theta)^k}{\nu(1 - \theta)^{\nu-1}} = \frac{1}{\nu} \sum_{k=0}^{\nu-1} (1 - \theta)^{-k},$$

the last step by reversing the order of summation. The right side is the average of the first ν terms of $(1 - \theta)^{-k}$, a sequence that is strictly increasing in k and, for $k \geq 1$, in θ . Averaging in one more term of a strictly increasing sequence raises the average, so the average exceeds its first term 1 and is strictly increasing in ν ; it is strictly increasing in θ as well. Inverting gives the claims about η_ν , and $\chi = \eta_m(\alpha)$ because $\psi_m(\alpha) = m\theta^*$ and $\alpha \psi'_m(\alpha) = m\underline{\theta}$. ■

Lemma C.2. As α rises on $(0, 1]$, Υ increases strictly, from $\Upsilon(0^+) = 0$ to $\Upsilon(1) > 1$, while $(m-1)/(m-\eta^*) \in (0, 1)$ decreases strictly. Hence there is a unique $\bar{\alpha} \in (0, 1)$ with $V^C(\bar{\alpha}) = V^M(\bar{\alpha})$, and $V^C(\alpha) > V^M(\alpha)$ iff $\alpha > \bar{\alpha}$.

Proof. An integral form for Υ . Writing $\psi^* - \psi(\underline{\theta}) = \int_{\underline{\theta}}^{\theta^*} \psi'(s) ds$ and substituting $s = v\theta^*$, with $\underline{\theta} = \chi\theta^*$ and $\psi'(s) = n(1-s)^{n-1}$,

$$\Upsilon = \int_{\chi}^1 \frac{\psi'(v\theta^*)}{\psi'(\theta^*)} dv = \int_{\chi}^1 \left(1 + \frac{(1-v)\theta^*}{1-\theta^*}\right)^{n-1} dv. \quad (\text{C.5})$$

The integrand exceeds 1 for $v < 1$ and is strictly increasing in θ^* and in n .

Monotonicity. Since $\theta^* = [1 - (1-\alpha)^m]/m$ is strictly increasing in α , $\eta^* = \eta_m(\theta^*) \in (0, 1)$ is strictly decreasing by Lemma C.1, and so is $(m-1)/(m-\eta^*) \in (0, 1)$. The same lemma makes χ strictly decreasing, so in (C.5) both the integrand and the range of integration grow with α , and Υ is strictly increasing.

Endpoints. As $\alpha \downarrow 0$, $\chi = \eta_m(\alpha) \rightarrow 1$ by (C.4) while $\theta^* \rightarrow 0$, so $\Upsilon \rightarrow 0$. At $\alpha = 1$, $\underline{\theta} = 0$, so $\chi = 0$ and $\Upsilon > 1$. Thus $\Upsilon - (m-1)/(m-\eta^*)$ is continuous and strictly increasing, negative near 0 and positive at 1, so it has a unique zero $\bar{\alpha} \in (0, 1)$, which by (C.2) is the unique zero of $V^C - V^M$. ■

Comparative statics of $\bar{\alpha}$ in n . Write $\bar{\alpha}_{m,n}$ for the cutoff of the $m \times n$ market.

Lemma C.3. $\bar{\alpha}_{m,n}$ is strictly decreasing in n .

Proof. Fix m and α ; neither θ^* nor χ depends on n . By Lemma C.1, $\eta^* = \eta_m(\theta^*)$ is strictly decreasing in n , and so is $(m-1)/(m-\eta^*)$, while Υ is strictly increasing in n by (C.5). So at $\alpha = \bar{\alpha}_{m,n}$, where the two sides of (C.2) are equal with n buyers, Υ is the larger side with $n+1$ buyers, and $\bar{\alpha}_{m,n+1} < \bar{\alpha}_{m,n}$ by Lemma C.2. ■

Both channels are intuitive: a larger n makes a seller's trading probability less elastic, which softens price competition and raises p^* , and it widens the trading-probability advantage Υ of competing over serving captives; the local-monopoly deviation loses on both counts.

The large-market cutoffs $\bar{\mu}$ and $\hat{\mu}$. Recall from Section 3 that $\varepsilon = \tau(1 - e^{-\mu})$, $V^C = 1 - (1 + \varepsilon)e^{-\varepsilon}$ and $V^M = 1 - e^{-\varepsilon_1}$ with $\varepsilon_1 = \tau\mu e^{-\mu}$, that $\bar{\mu}$ separates the pure- from the mixed-strategy region, and that $\hat{\mu}$ marks the appearance of the atom, $\kappa > 0$. Two facts from the proof of Proposition 3 are used here and in Appendix E. First,

the tightness at the atom is pinned down by V^M alone: since $z = \ln(1 - V^M/\hat{p}) - 1 = -(1 + \varepsilon_{\hat{p}})$, the condition $ze^z = (V^M - 1)/e$ defining $\hat{p}$ reads $(1 + \varepsilon_{\hat{p}})e^{-\varepsilon_{\hat{p}}} = 1 - V^M$, or

$$\varepsilon_{\hat{p}} - \ln(1 + \varepsilon_{\hat{p}}) = \varepsilon_1, \quad (\text{C.6})$$

which has a unique solution because $x - \ln(1 + x)$ is strictly increasing on $x > 0$. Second, $\kappa > 0$ iff $\Gamma_{\hat{p}} > 1$, that is, $\varepsilon_{\hat{p}} < \tau\mu$. Both cutoffs are governed by

$$A(x) \equiv 1 - \frac{\ln(1 + x)}{x} \quad \text{and} \quad B(\mu) \equiv \frac{\mu}{e^\mu - 1}, \quad x, \mu > 0. \quad (\text{C.7})$$

Since $\ln(1 + x)/x$ is the average of the strictly decreasing function $(1 + t)^{-1}$ over $[0, x]$, A is strictly increasing with $A(0^+) = 0$ and $A(\infty) = 1$. Since $(e^\mu - 1)/\mu$ is the average of e^t over $[0, \mu]$, B is strictly decreasing with $B(0^+) = 1$ and $B(\infty) = 0$; and $e^{-\mu} < B(\mu)$ for every $\mu > 0$, as that average is strictly below its largest value e^μ . Both functions have a reading in the model. Since $xA(x) = -\ln[(1 + x)e^{-x}]$, $\varepsilon A(\varepsilon) = -\ln(1 - V^C)$ is the queue at which a seller sells with probability V^C , just as $\varepsilon_1 = -\ln(1 - V^M)$ is the queue at $p = 1$. And B is the elasticity of the urn-ball matching function $1 - e^{-x}$, the large-market counterpart of η : $B(\varepsilon) = 1 - p^*$ by (8), and $B(\mu) = \varepsilon_1/\varepsilon$ is the large-market captive share, the counterpart of $\chi = \eta_m(\alpha)$.

Lemma C.4. With $\varepsilon = \tau(1 - e^{-\mu})$,

$$V^C \geq V^M \iff A(\varepsilon) - B(\mu) \geq 0, \quad \kappa > 0 \iff A(\tau\mu) - e^{-\mu} > 0. \quad (\text{C.8})$$

Both expressions in (C.8) are continuous and strictly increasing in μ , tend to -1 as $\mu \downarrow 0$ and to a positive limit as $\mu \rightarrow \infty$, and are strictly increasing in τ at every $\mu > 0$. Hence $\bar{\mu}$ and $\hat{\mu}$ are their unique zeros, both are strictly decreasing in τ , and $\bar{\mu} > \hat{\mu}$ for every $\tau < \infty$.

Proof. Reductions. $V^C \geq V^M$ iff $(1 + \varepsilon)e^{-\varepsilon} \leq e^{-\varepsilon_1}$, that is, $\varepsilon - \ln(1 + \varepsilon) \geq \varepsilon_1$; dividing by $\varepsilon > 0$ turns the left side into $A(\varepsilon)$ and the right side into $\varepsilon_1/\varepsilon = B(\mu)$. Next, $\kappa > 0$ iff $\varepsilon_{\hat{p}} < \tau\mu$; as $x - \ln(1 + x)$ is increasing, by (C.6) this holds iff $\varepsilon_1 < \tau\mu - \ln(1 + \tau\mu)$, that is, dividing by $\tau\mu$, $e^{-\mu} < A(\tau\mu)$.

Monotonicity and endpoints. ε and $\tau\mu$ are strictly increasing in μ and in τ , A is strictly increasing, and $B(\mu)$ and $e^{-\mu}$ are strictly decreasing, so both expressions in (C.8) are strictly increasing in μ and in τ . As $\mu \downarrow 0$, both A terms vanish while

$B(\mu) \rightarrow 1$ and $e^{-\mu} \rightarrow 1$, so both expressions tend to -1 ; as $\mu \rightarrow \infty$, the first tends to $A(\tau) > 0$ and the second to 1. Each expression therefore has a unique zero, $\bar{\mu}$ and $\hat{\mu}$ respectively, and is negative below it and positive above it; a larger τ makes each expression positive at its old zero, so $\bar{\mu}$ and $\hat{\mu}$ are strictly decreasing in τ .

Ranking. Fix $\tau < \infty$ and $\mu > 0$. Because $\mu > 1 - e^{-\mu}$ implies $\tau\mu > \varepsilon$, $A(\tau\mu) > A(\varepsilon)$; and $e^{-\mu} < B(\mu)$. Adding the two strict inequalities, $A(\tau\mu) - e^{-\mu} > A(\varepsilon) - B(\mu)$. At $\mu = \bar{\mu}$ the right side vanishes, so the left side is positive. Hence $\hat{\mu} < \bar{\mu}$. ■

The two cutoffs merge only in the limit: at any fixed $\mu > 0$, as $\tau \rightarrow \infty$ the two expressions in (C.8) converge to the positive limits $1 - B(\mu)$ and $1 - e^{-\mu}$, placing μ above both cutoffs. Hence $\bar{\mu}$ and $\hat{\mu}$ vanish as $\tau \rightarrow \infty$.

Comparative statics of $\bar{\alpha}$ in m . Return to the finite market, let $\tilde{\mu} \equiv -m \ln(1 - \alpha)$ be the finite-market version of μ , chosen so that the idle share is $(1 - \alpha)^m = e^{-\tilde{\mu}}$ as in Case 2, and write $\varepsilon \equiv n\theta^* = \tau(1 - e^{-\tilde{\mu}})$ for the effective tightness, the Case 2 formula with $\tilde{\mu}$ in place of μ . Then $\theta^* = (1 - e^{-\tilde{\mu}})/m$, and $\chi = B(\tilde{\mu})/B(\tilde{\mu}/m)$ falls in m to $B(\tilde{\mu})$ at fixed $\tilde{\mu}$. Moreover, B is $\frac{1}{2}$ -Lipschitz, since $-B'(x) \leq \frac{1}{2}$ reduces to $\sinh x \geq x$, so $1 - B(\tilde{\mu}) \leq \tilde{\mu}/2$. Finally, as $\eta^* = \varepsilon[(1 - \theta^*)^{-n} - 1]^{-1}/(1 - \theta^*)$ and $e^\varepsilon \leq (1 - \theta^*)^{-n} \leq e^{\varepsilon/(1 - \theta^*)}$, the elasticity is close to its large-market value $B(\varepsilon)$: $B(\varepsilon/(1 - \theta^*)) \leq \eta^* \leq B(\varepsilon)/(1 - \theta^*)$.

Lemma C.5. $\bar{\alpha}_{m,n}$ is strictly decreasing in m .

Proof. Reduction. Treat $m \geq 2$ as real. Lemmas C.2 and C.3 still hold, because $1/\chi = [(1 - \alpha)^{-m} - 1]/[m(1 - \alpha)^{-1} - m]$ is the average of y^{m-1} over $y \in [1, (1 - \alpha)^{-1}]$ and so still rises from 1 with α . At the cutoff, $\partial_\alpha \ln(V^C/V^M) > 0$: by the derivation of (C.2), $V^C - V^M = \theta^* \psi'(\theta^*) [\Upsilon - (m - 1)/(m - \eta^*)]$, and the bracket vanishes at the cutoff and rises with α , as Υ does and $\partial_\alpha \eta^* = \eta'(\theta^*) \partial_\alpha \theta^* < 0$. By the implicit function theorem, $\bar{\alpha}$ is therefore C^1 in m , and $\bar{\alpha}' < 0$ iff, at $\alpha = \bar{\alpha}$, an extra seller hurts the local monopolist more than a competing seller: $-\partial \ln V^M / \partial \ln m > -\partial \ln V^C / \partial \ln m$. At fixed α , $\partial \ln \underline{\theta} / \partial \ln m = -\tilde{\mu}$ and $\partial \ln \theta^* / \partial \ln m = -[1 - B(\tilde{\mu})]$, so, as $V^M = \psi(\underline{\theta})$ and $V^C = \psi^* m(1 - \eta^*)/(m - \eta^*)$, the two payoffs fall at the rates

$$\begin{aligned} -\frac{\partial \ln V^M}{\partial \ln m} &= \tilde{\mu} \eta(\underline{\theta}), \\ -\frac{\partial \ln V^C}{\partial \ln m} &= [1 - B(\tilde{\mu})] \eta^* + \frac{\eta^*}{m - \eta^*} + [1 - B(\tilde{\mu})] \frac{m - 1}{m - \eta^*} \cdot \frac{-\theta^* \eta'(\theta^*)}{1 - \eta^*}. \end{aligned} \quad (\text{C.9})$$

An extra seller cuts the captive queue by the factor $1 - \alpha = e^{-\tilde{\mu}/m}$, since a captive buyer must now also miss it, whereas the competing queue shrinks at most half as fast because the new seller also draws in idle buyers; V^C falls further through p^* , with one more rival and more elastic sales.

The criterion at the cutoff. Log-differentiating $\eta = \theta\psi'/\psi$, with $\psi'' = -(n-1)\psi'/(1-\theta)$, gives $\theta\eta'/\eta = 1 - \eta - (n-1)\theta/(1-\theta)$. Let $\zeta \equiv (1-\underline{\theta})/(1-\theta^*)$, so that $\zeta^n = (1-V^M)/(1-\psi^*)$ is the ratio of a seller's no-sale probabilities when serving only captives and when competing, $\zeta - 1 = (\theta^* - \underline{\theta})/(1-\theta^*) = (1-\chi)\theta^*/(1-\theta^*)$, and $\Upsilon = (1-\theta^*)(\zeta^n - 1)/(n\theta^*)$ by (C.1). At the cutoff, (C.2) then gives

$$\zeta^n = 1 + \frac{(m-1)\varepsilon}{(1-\theta^*)(m-\eta^*)}, \quad (\text{C.10})$$

and $\eta(\underline{\theta})/\eta^* = \chi\zeta^{n-1}\psi^*/\psi(\underline{\theta}) = \chi\zeta^{n-1}/p^*$. Substituting these into (C.9) and multiplying by $(m-\eta^*)(1-\eta^*)/\eta^*$, the criterion becomes

$$\begin{aligned} & \frac{\tilde{\mu}\chi(m-\eta^*)}{m(1-\underline{\theta})} [(m-\eta^*)(1-\theta^*) + (m-1)\varepsilon] \\ & > (1-\eta^*) + [1 - B(\tilde{\mu})] \left[(1-\eta^*)^2 + \frac{(m-1)(n-1)\theta^*}{1-\theta^*} \right]. \end{aligned} \quad (\text{C.11})$$

The criterion is verified in three cases.

Few buyers per seller, $m \geq 4n$. Here $\theta^* \leq 1/m$ and $(1-\theta^*)^{1-n} \leq e^{(n-1)/(m-1)} < e^{1/4}$. As $1-\eta^*$ is the probability that a seller with a visitor has at least two, and visitors are binomial with parameters n and θ^* , $\frac{1}{2}(n-1)\theta^*(1-\theta^*)^{n-2} \leq 1-\eta^* \leq \frac{1}{2}(n-1)\theta^*(1-\theta^*)^{1-n}$. By the lower bound, $-\theta^*\eta'(\theta^*)/[\eta^*(1-\eta^*)] = (n-1)\theta^*/[(1-\theta^*)(1-\eta^*)] - 1 < 2e^{1/4} - 1$, so the rate of V^C in (C.9) is at most $\eta^*[1/(m-1) + 2e^{1/4}] < 2.72\eta^*$, as $m \geq 8$. By the concavity of ψ , $\psi(\underline{\theta}) \geq \chi\psi^*$, so at the cutoff $\chi \leq p^* \leq m(1-\eta^*)/(m-1)$, which the upper bound places below $(n-1)e^{1/4}/[2(m-1)] < e^{1/4}/8 < B(2.72)$. Hence $\tilde{\mu} > 2.72$, as $B(\tilde{\mu}) \leq \chi$, and the rate of V^M is $\tilde{\mu}\eta(\underline{\theta}) > 2.72\eta^*$, as $\eta(\cdot)$ is decreasing.

Small samples, $\tilde{\mu} \leq 2/7$. Here $\underline{\theta} < \theta^* \leq \tilde{\mu}/m \leq 1/7$, $\chi \geq B(\tilde{\mu}) \geq 6/7$ and, as B is $\frac{1}{2}$ -Lipschitz and $1/B(x) < e^x$, $1-\chi = [B(\tilde{\mu}/m) - B(\tilde{\mu})]/B(\tilde{\mu}/m) \leq (1-1/m)\tilde{\mu}e^{\tilde{\mu}}/2$. Let $D \equiv (1-1/m)\varepsilon$. In (C.10), $n \ln \zeta \leq n(\zeta - 1) = \varepsilon(1-\chi)/(1-\theta^*)$, while the right side is at least $1 + D$. Hence $\ln(1 + D) \leq \frac{7}{12}\tilde{\mu}e^{\tilde{\mu}}D \leq \frac{1}{6}e^{2/7}D$, which forces $D > 11.3$: sellers almost surely trade. Then the elasticity bounds give $\eta^* \leq \frac{7}{6}B(11.3) < 2 \cdot 10^{-4}$, and $\tilde{\mu}D \geq \frac{12}{7}e^{-\tilde{\mu}}\ln(1 + D) > 3.2$. In (C.11), where $(m-1)\varepsilon = mD$, the left side is

thus at least $0.85\tilde{\mu}(1+mD)$, as $\chi(1-\eta^*/m) > 0.85$ and $(m-\eta^*)(1-\theta^*) > 1$, and the right side at most $1 + \frac{\tilde{\mu}}{2}(1 + \frac{7}{6}mD)$, as $(n-1)\theta^* \leq \varepsilon$ and $1 - B(\tilde{\mu}) \leq \tilde{\mu}/2$; the difference exceeds $0.26m\tilde{\mu}D - 1 > 0$.

Many sellers, $m \geq 20$, $n \geq m/4$ and $\tilde{\mu} > 2/7$. Let $\hat{\theta} \equiv (1 - e^{-\tilde{\mu}})/20$ and $\hat{\chi} \equiv B(\tilde{\mu})/B(\tilde{\mu}/20)$ be the values of θ^* and χ at $m = 20$, so that $\theta^* \leq \hat{\theta}$ and $B(\tilde{\mu}) \leq \chi \leq \hat{\chi}$. In (C.10), $n \ln \zeta \geq n(\theta^* - \underline{\theta}) = \varepsilon(1 - \chi)$ and the right side is at most $1 + \varepsilon/(1 - \hat{\theta})$, while $n \geq m/4$ gives $\varepsilon \geq (1 - e^{-\tilde{\mu}})/4$, so

$$\varepsilon(1 - \hat{\chi}) \leq \ln\left(1 + \frac{\varepsilon}{1 - \hat{\theta}}\right) \quad \text{and} \quad \varepsilon \geq \frac{1 - e^{-\tilde{\mu}}}{4}. \quad (\text{C.12})$$

In (C.11), use $\chi \geq B(\tilde{\mu})$, $1 - \underline{\theta} \leq 1$, $m - \eta^* \geq m - 1$, $1 - \eta^*/m \geq 19/20$, $\theta^* \leq \hat{\theta}$, $(n-1)\theta^* \leq \varepsilon$ and, by the elasticity bounds, $\eta^* \geq \hat{\eta} \equiv B(\varepsilon/(1 - \hat{\theta}))$. Dividing by $m - 1 \geq 19$, (C.11) holds whenever

$$\frac{19}{20}\tilde{\mu}B(\tilde{\mu})(1 - \hat{\theta} + \varepsilon) > \frac{1 - \hat{\eta} + [1 - B(\tilde{\mu})](1 - \hat{\eta})^2}{19} + \frac{[1 - B(\tilde{\mu})]\varepsilon}{1 - \hat{\theta}}. \quad (\text{C.13})$$

Both conditions involve only $(\tilde{\mu}, \varepsilon)$, and a numerical evaluation shows that (C.13) holds, with a ratio of the two sides of at least 1.13, at every $(\tilde{\mu}, \varepsilon)$ with $\tilde{\mu} \geq 2/7$ that satisfies (C.12); these lie within $\tilde{\mu} < 4.33$ and $\varepsilon < 26$. The margin reflects the large-market limit: with 20 replaced by ∞ , (C.12) and (C.13) become $A(\varepsilon) \leq B(\tilde{\mu})$ and $\tilde{\mu}B(\tilde{\mu})(1 + \varepsilon) > [1 - B(\tilde{\mu})]\varepsilon$, and the first implies the second, as $\tilde{\mu} \geq 2[1 - B(\tilde{\mu})]$ and $A(\varepsilon) > \varepsilon/[2(1 + \varepsilon)]$ by the trapezoid rule for $\ln(1 + \varepsilon) = \int_0^\varepsilon (1 + t)^{-1} dt$, whose integrand is convex.

Conclusion. For real $m \geq 20$, every cutoff falls in one of the three cases, so $\bar{\alpha}$ is strictly decreasing on $[20, \infty)$. For $m \leq 19$, two numerical facts complete the proof. First, at $n = 3200$ the cutoff has $\tilde{\mu} < 2/7$ for every real $m \in [2, 20]$; as $\bar{\alpha}$ falls in n and $\tilde{\mu}$ rises in α , so does every cutoff with $n \geq 3200$, and the case $\tilde{\mu} \leq 2/7$ makes $\bar{\alpha}$ strictly decreasing on $[2, 20]$. Second, direct computation gives $\bar{\alpha}_{m+1,n} < \bar{\alpha}_{m,n}$ for $2 \leq m \leq 19$ and $2 \leq n \leq 3199$. ■

D Details for the $m = 2$ case

Fix $\alpha < \bar{\alpha}$ and take $m = 2$. By the arguments of Huang (2016), with a single rival no two atoms of $\text{supp}(F^*)$ locally compete: a seller's queue depends only on whether it

is cheaper than, tied with, or pricier than its rival, and a small change in its own price does not alter this ranking, as the atoms constructed below are far enough apart. The support is therefore a sequence of atoms $p_1 < p_2 < \dots$, infinite and accumulating at 1, and every seller earns V^M (Proposition 1).

With two sellers, $\underline{\theta} = \alpha(1 - \alpha) < \theta^* < \alpha$, so the selling probabilities

$$\bar{\psi} \equiv \psi(\alpha), \quad \psi^* \equiv \psi(\theta^*), \quad \underline{\psi} \equiv \psi(\underline{\theta}) = V^M \quad (\text{D.1})$$

of a seller that is respectively cheaper than, tied with, and pricier than its rival satisfy $\bar{\psi} > \psi^* > \underline{\psi} > 0$.

The difference equation. Write $\kappa_k \equiv F^*(\{p_k\})$ for the mass of the k -th atom, so that $F^*(p_k) = \sum_{i \leq k} \kappa_i$. A seller posting p_k meets a cheaper atom with probability $F^*(p_k) - \kappa_k$, a tie with probability κ_k , and a pricier atom with probability $1 - F^*(p_k)$, so the equal-profit condition $V(p_k; F^*) = \underline{\psi}$ reads

$$p_k \left[\bar{\psi} - (\bar{\psi} - \underline{\psi}) F^*(p_k) + (\psi^* - \underline{\psi}) \kappa_k \right] = \underline{\psi} \quad (\text{D.2})$$

for every k . Because no atom locally competes with another, a small change in a seller's own price away from p_k leaves its queues against cheaper and pricier rivals untouched and moves only the tie, in which the two sellers form a block; by (B.3) with $\beta_j = 2$, its queue in the tie falls at the rate $\varrho(\theta^*)/[2(1 - p_k)] = \theta^*/[2(1 - p_k)(1 - \eta^*)]$. Optimality of p_k therefore reads

$$\underline{\psi} = p_k^2 \kappa_k \psi'(\theta^*) \frac{\theta^*}{2(1 - p_k)(1 - \eta^*)}. \quad (\text{D.3})$$

Substituting $\theta^* \psi'(\theta^*) = \eta^* \psi^*$ and $(1 - p^*)/p^* = \eta^*/[2(1 - \eta^*)]$ from (5) at $m = 2$, the mass of each atom is pinned down by its price alone:

$$\kappa_k = \frac{\underline{\psi}(1 - p_k)}{\lambda p_k^2}, \quad \lambda \equiv \frac{\psi^*(1 - p^*)}{p^*} > 0. \quad (\text{D.4})$$

Subtracting (D.2) at consecutive indices, using $F^*(p_{k+1}) = F^*(p_k) + \kappa_{k+1}$, and substituting (D.4) gives the difference equation

$$\frac{1}{p_{k+1}} + \frac{(\bar{\psi} - \psi^*)(1 - p_{k+1})}{\lambda p_{k+1}^2} = \frac{1}{p_k} - \frac{(\psi^* - \underline{\psi})(1 - p_k)}{\lambda p_k^2}, \quad (\text{D.5})$$

with the initial condition obtained from (D.2) at $k = 1$, where $F^*(p_1) = \kappa_1$,

$$\frac{1}{p_1} + \frac{(\bar{\psi} - \psi^*)(1 - p_1)}{\lambda p_1^2} = \frac{\bar{\psi}}{\underline{\psi}}. \quad (\text{D.6})$$

Closed form. In terms of $v \equiv 1/p$, for which $(1 - p)/p^2 = v(v - 1)$, both sides of (D.5) are quadratic. Set $b_+ \equiv (\bar{\psi} - \psi^*)/\lambda$, $b_- \equiv (\psi^* - \underline{\psi})/\lambda$, $\Lambda_+(v) \equiv v + v(v - 1)b_+$ and $\Lambda_-(v) \equiv v - v(v - 1)b_-$, so that (D.5) reads $\Lambda_+(v_{k+1}) = \Lambda_-(v_k)$ and (D.6) reads $\Lambda_+(v_1) = \bar{\psi}/\underline{\psi}$. Both maps fix $v = 1$, and Λ_+ is a strictly increasing bijection of $[1, \infty)$ onto itself with

$$\Lambda_+^{-1}(y) = \frac{-(1 - b_+) + \sqrt{(1 - b_+)^2 + 4b_+y}}{2b_+}. \quad (\text{D.7})$$

The atoms then follow from the explicit recursion

$$v_1 = \Lambda_+^{-1}\left(\frac{\bar{\psi}}{\underline{\psi}}\right), \quad v_{k+1} = (\Lambda_+^{-1} \circ \Lambda_-)(v_k), \quad p_k = \frac{1}{v_k}, \quad (\text{D.8})$$

and the masses from (D.4), $\kappa_k = v_k(v_k - 1)\underline{\psi}/\lambda$.

Consistency. The recursion has a simple reading. By (D.4), $\underline{\psi}\Lambda_-(v_k) = \underline{\psi}v_k - (\psi^* - \underline{\psi})\kappa_k$, and (D.2) turns the right side into $\bar{\psi} - (\bar{\psi} - \underline{\psi})F^*(p_k)$, so

$$\Lambda_-(v_k) - 1 = \frac{(\bar{\psi} - \underline{\psi})[1 - F^*(p_k)]}{\underline{\psi}}, \quad (\text{D.9})$$

that is, the input to the next step exceeds 1 by a multiple of the mass not yet assigned. Since $\Lambda_-(v) - 1 = (v - 1)(1 - b_-v)$, for $v_k > 1$ the conditions $F^*(p_k) < 1$, $\Lambda_-(v_k) > 1$ and $v_k < 1/b_-$ coincide, and they keep v_{k+1} in $(1, \infty)$, that is, $p_{k+1} \in (0, 1)$.

At $k = 1$, $v_1 > 1$ because $\bar{\psi}/\underline{\psi} > 1$. By (D.2) at $k = 1$, $\kappa_1 < 1$ iff $\underline{\psi}v_1 > \psi^*$, and since Λ_+ is increasing with $\Lambda_+(v_1) = \bar{\psi}/\underline{\psi}$, this is

$$\frac{\bar{\psi}}{\underline{\psi}} > \Lambda_+\left(\frac{\psi^*}{\underline{\psi}}\right) = \frac{\psi^*}{\underline{\psi}} + \frac{(\bar{\psi} - \psi^*)\psi^*(\psi^* - \underline{\psi})}{\lambda \underline{\psi}^2},$$

which simplifies to $\lambda \underline{\psi} > \psi^*(\psi^* - \underline{\psi})$, that is, by the definition of λ , to $\underline{\psi} > p^*\psi^*$. The final inequality is $V^M > V^C$, which holds for $\alpha < \bar{\alpha}$; at $\alpha = \bar{\alpha}$ it becomes an equality, $\kappa_1 = 1$ and $p_1 = \underline{\psi}/\psi^* = p^*$, so the mixed equilibrium collapses onto the pure one at

the cutoff.

Now suppose $1 < v_k < 1/b_-$, as holds at $k = 1$. Then $\Lambda_-(v_k) > 1$, and $1 < v_{k+1} < \Lambda_-(v_k) < v_k < 1/b_-$, since $\Lambda_-(v) < v < \Lambda_+(v)$ for $v > 1$, so that $\Lambda_+^{-1}(y) < y$ for $y > 1$. By induction, v_k is strictly decreasing and stays in $(1, 1/b_-)$: every atom lies in $(0, 1)$, every κ_k is positive, and by (D.9) every $F^*(p_k)$ is below 1. Being bounded below by 1, v_k converges to a solution of $\Lambda_+(v) = \Lambda_-(v)$, that is, of $(b_+ + b_-)v(v - 1) = 0$, so $v_k \downarrow 1$ and the atoms increase to 1. Moreover, atoms do not locally compete, $(1 - p_{k+1})\underline{\psi} \leq (1 - \alpha)(1 - p_k)\bar{\psi}$, in every case computed. Finally, $\Lambda_-(v_k) \rightarrow 1$ in (D.9) gives $F^*(p_k) \rightarrow 1$: the masses sum to one and F^* is a proper distribution.

E Proof of Proposition 4

Throughout, τ is fixed and suppressed as an argument, so that $V^M(1)$ denotes $V^M(1, \tau)$; the three statements are proved in order.

The rank representation. Let $\mu < \bar{\mu}$ and index the interval $[\underline{p}, 1]$ in $\text{supp}(F^*)$ by the rank $s = F^*(p)$. By (13), a seller at rank $s \in [\kappa, 1]$ faces the queue

$$\varepsilon(s) = \tau \mu e^{-\mu s}, \quad (\text{E.1})$$

so $\varepsilon(\kappa) = \varepsilon_{\underline{p}}$ as in (16) and $\varepsilon(1) = \varepsilon_1$. The atom occupies the ranks $[0, \kappa)$ and, by (19), faces

$$\hat{\varepsilon}(\kappa) \equiv \frac{\tau(1 - e^{-\mu\kappa})}{\kappa} = \frac{1}{\kappa} \int_0^\kappa \varepsilon(s) \, ds, \quad (\text{E.2})$$

the average of (E.1) over the ranks it spans: the atom flattens the queue. Treating κ as a free argument, define

$$\tilde{\Psi}(\mu, \kappa) \equiv \kappa (1 - e^{-\hat{\varepsilon}(\kappa)}) + \int_\kappa^1 (1 - e^{-\varepsilon(s)}) \, ds, \quad (\text{E.3})$$

which returns the Ψ^* of (20) at the equilibrium κ of (19), where $\hat{\varepsilon}(\kappa) = \varepsilon_{\hat{p}}$. The equilibrium κ is below 1, as $h(1) = \varepsilon - \varepsilon_{\hat{p}} < 0$ (proof of Proposition 3). At $\mu = \bar{\mu}$, instead, $\varepsilon_{\hat{p}} = \varepsilon$, so $\kappa = 1$ and (E.3) equals $\Psi^{(2)}$: Ψ^* is continuous at $\bar{\mu}$.

(i) *The Jensen gap.* By (E.2), the mean queue across ranks is $\kappa \hat{\varepsilon}(\kappa) + \int_\kappa^1 \varepsilon(s) \, ds = \int_0^1 \varepsilon(s) \, ds = \varepsilon$ for every κ : pricing reallocates the queue without changing its total.

Since $1 - e^{-x}$ is strictly concave, Jensen's inequality gives $\Psi^* \leq \Psi^{(2)}$, with equality only if the queue is constant across ranks, that is, $\kappa = 1$. Hence $\Psi^* < \Psi^{(2)}$ for $\mu < \bar{\mu}$ and $\Psi^* = \Psi^{(2)}$ for $\mu \geq \bar{\mu}$; and $\Psi^{(2)} < \min\{1, \varepsilon\} \leq \min\{1, \tau\} = \Psi^{(1)}$.

(i) *Monotonicity.* For $\mu \geq \bar{\mu}$, $\Psi^* = \Psi^{(2)}$ is strictly increasing because ε is. For $\mu < \bar{\mu}$, $d\Psi^*/d\mu = \partial_\mu \tilde{\Psi} + \partial_\kappa \tilde{\Psi} \partial_\mu \kappa$, and the second term is present only on $(\hat{\mu}, \bar{\mu})$, where $\kappa > 0$. The three factors are signed in turn.

(a) $\partial_\kappa \tilde{\Psi} > 0$. Differentiating $\kappa \hat{\varepsilon} = \tau(1 - e^{-\mu\kappa})$ gives $\hat{\varepsilon} + \kappa \hat{\varepsilon}' = \varepsilon(\kappa)$, so

$$\begin{aligned} \partial_\kappa \tilde{\Psi} &= (1 - e^{-\hat{\varepsilon}}) + \kappa e^{-\hat{\varepsilon}} \hat{\varepsilon}' - (1 - e^{-\varepsilon(\kappa)}) \\ &= e^{-\varepsilon(\kappa)} - [e^{-\hat{\varepsilon}} - e^{-\hat{\varepsilon}} (\varepsilon(\kappa) - \hat{\varepsilon})] > 0, \end{aligned} \quad (\text{E.4})$$

the bracket being the tangent line to the strictly convex e^{-x} at $\hat{\varepsilon}$, evaluated at $\varepsilon(\kappa) < \hat{\varepsilon}$, as $\hat{\varepsilon}$ averages the decreasing $\varepsilon(s)$ over $[0, \kappa]$. Enlarging the atom contracts the spread of queues at a fixed mean, and trade is concave in the queue.

(b) $\partial_\mu \kappa > 0$. As $\partial_\kappa h < 0$ at the positive root, $\partial_\mu \kappa$ has the sign of $\partial_\mu h$. Differentiating (C.6) gives $\partial_\mu \varepsilon_{\hat{p}} = (1 + \varepsilon_{\hat{p}})\tau e^{-\mu}(1 - \mu)/\varepsilon_{\hat{p}}$, so

$$\partial_\mu h = \kappa \tau e^{-\mu\kappa} - \kappa \partial_\mu \varepsilon_{\hat{p}} = \frac{\kappa \tau e^{-\mu}}{\varepsilon_{\hat{p}}} [\varepsilon_{\hat{p}} e^{\mu(1-\kappa)} - (1 + \varepsilon_{\hat{p}})(1 - \mu)].$$

For $\mu \geq 1$ the bracket is positive. For $\mu < 1$, as $e^{\mu(1-\kappa)} > 1$, it suffices that $\mu \varepsilon_{\hat{p}} \geq 1 - \mu$. Now $\kappa > 0$ means $\varepsilon_{\hat{p}} < \tau\mu$, so by (C.6) $\varepsilon_{\hat{p}} - \ln(1 + \varepsilon_{\hat{p}}) = \tau\mu e^{-\mu} > \varepsilon_{\hat{p}} e^{-\mu}$, that is, $\ell(\varepsilon_{\hat{p}}) > 0$ for the convex function $\ell(x) \equiv x(1 - e^{-\mu}) - \ln(1 + x)$, which vanishes at 0. At $x_0 \equiv (1 - \mu)/\mu$, using $1 - e^{-\mu} \leq \mu$ and $\ln \mu \leq \mu - 1$,

$$\ell(x_0) = x_0 (1 - e^{-\mu}) + \ln \mu \leq (1 - \mu) + \ln \mu \leq 0,$$

so $\ell \leq 0$ on $[0, x_0]$ by convexity, and hence $\varepsilon_{\hat{p}} > x_0$, that is, $\mu \varepsilon_{\hat{p}} > 1 - \mu$.

(c) $\partial_\mu \tilde{\Psi} > 0$. At fixed κ , $\partial_\mu \hat{\varepsilon} = \varepsilon(\kappa)/\mu$ and $\partial_\mu \varepsilon(s) = \varepsilon(s)(1 - \mu s)/\mu$, so

$$\mu \partial_\mu \tilde{\Psi} = \kappa \varepsilon(\kappa) e^{-\hat{\varepsilon}} + \int_\kappa^1 \varepsilon(s) e^{-\varepsilon(s)} (1 - \mu s) ds. \quad (\text{E.5})$$

Since $\varepsilon'(s) = -\mu \varepsilon(s)$, the integrand satisfies

$$\varepsilon(s) e^{-\varepsilon(s)} (1 - \mu s) = (1 + \varepsilon(s)) e^{-\varepsilon(s)} - \frac{d}{ds} [s e^{-\varepsilon(s)}],$$

so integrating and splitting $e^{-\varepsilon(1)} = \kappa e^{-\varepsilon(1)} + \int_{\kappa}^1 e^{-\varepsilon(1)} ds$,

$$\mu \partial_{\mu} \tilde{\Psi} = \int_{\kappa}^1 [(1 + \varepsilon(s)) e^{-\varepsilon(s)} - e^{-\varepsilon(1)}] ds + \kappa [e^{-\varepsilon(\kappa)} + \varepsilon(\kappa) e^{-\hat{\varepsilon}} - e^{-\varepsilon(1)}]. \quad (\text{E.6})$$

The integral is positive: $(1 + x)e^{-x}$ is strictly decreasing in $x > 0$ and $\varepsilon(s)$ in s , so the integrand is strictly increasing in s ; as $\kappa < 1$, it suffices that the integrand is nonnegative at $s = \kappa$, that is,

$$\varepsilon(\kappa) - \ln(1 + \varepsilon(\kappa)) \leq \varepsilon_1. \quad (\text{E.7})$$

When $\kappa > 0$, $\varepsilon(\kappa) < \hat{\varepsilon} = \varepsilon_{\hat{\rho}}$, and (C.6) makes (E.7) hold strictly, as $x - \ln(1 + x)$ is increasing. When $\kappa = 0$, $\varepsilon(\kappa) = \tau\mu$ and (E.7) reads $A(\tau\mu) \leq e^{-\mu}$, which holds by (C.8). In the bracket, present only when $\kappa > 0$, (C.6) exponentiates to $e^{-\varepsilon(1)} = (1 + \hat{\varepsilon})e^{-\hat{\varepsilon}}$, so the bracket equals the expression in (E.4) and is positive by the same tangent-line argument. Hence $\partial_{\mu} \tilde{\Psi} > 0$ throughout the mixed region. Combining (a)–(c), Ψ^* is strictly increasing on $(0, \hat{\mu}]$, where $\kappa = 0$, and on $(\hat{\mu}, \bar{\mu})$; by continuity at $\hat{\mu}$, where $\kappa \rightarrow 0$, and at $\bar{\mu}$, it is strictly increasing on $(0, \infty)$. This is statement (i).

(ii) *Buyers.* On the pure-strategy region, $\tau U = \Psi^{(2)} - V^C = \varepsilon e^{-\varepsilon}$, so $U = (1 - e^{-\mu})e^{-\varepsilon}$ and $\partial_{\mu} U = e^{-\mu} e^{-\varepsilon} (1 - \varepsilon)$ has the sign of $1 - \varepsilon$. If $\tau \leq 1$, then $\varepsilon < \tau \leq 1$ and $\partial_{\mu} U > 0$ throughout. If $\tau > 1$, ε crosses 1 exactly at μ^* , so $\partial_{\mu} U$ changes sign from positive to negative at μ^* , and on $[\bar{\mu}, \infty)$, U is maximized at $\max\{\bar{\mu}, \mu^*\}$.

On the mixed-strategy region, $\tau U = \Psi^* - V^M$. For $\mu \geq 1$, V^M is nonincreasing while Ψ^* is strictly increasing by (i), so $\partial_{\mu} U > 0$. For $\mu < 1$ and $\tau \leq 1$, $\kappa = 0$: by (C.8), $\kappa > 0$ requires $A(\tau\mu) > e^{-\mu}$, whereas $A(\tau\mu) < A(1) = 1 - \ln 2 < e^{-1} < e^{-\mu}$. Writing $\xi(s) \equiv \varepsilon(s)e^{-\varepsilon(s)}$, (E.5) with $\kappa = 0$ and $V^M = 1 - e^{-\varepsilon(1)}$ give

$$\mu \partial_{\mu} \tilde{\Psi} = \int_0^1 \xi(s) (1 - \mu s) ds, \quad \mu \partial_{\mu} V^M = (1 - \mu) \xi(1).$$

Since $\varepsilon(s) \leq \tau\mu < 1$, $\xi'(s) = -\mu \xi(s) (1 - \varepsilon(s)) < 0$, so ξ is strictly decreasing and

$$\mu \partial_{\mu} \tilde{\Psi} \geq (1 - \mu) \int_0^1 \xi(s) ds > (1 - \mu) \xi(1) = \mu \partial_{\mu} V^M,$$

so $\partial_{\mu} U > 0$. Since U is continuous at $\bar{\mu}$, where Ψ^* is continuous and $V^M = V^C$, the pieces combine: for $\tau \leq 1$, U is strictly increasing on $(0, \infty)$; for $\tau > 1$, it is strictly in-

creasing on $[\min\{1, \bar{\mu}\}, \max\{\bar{\mu}, \mu^*\}]$ and strictly decreasing on $[\max\{\bar{\mu}, \mu^*\}, \infty)$, hence single-peaked on $[\min\{1, \bar{\mu}\}, \infty)$. This is statement (ii).

(iii) *Sellers*. By (10), V^M is strictly increasing in $\mu e^{-\mu}$, which peaks at $\mu = 1$; this is the first part of (iii). On $[\bar{\mu}, \infty)$, $V = V^C$ is strictly increasing in ε and hence in μ , with supremum $\bar{V} \equiv 1 - (1 + \tau)e^{-\tau}$, the competing payoff of Case 1, approached only as $\mu \rightarrow \infty$. The only candidates for a maximum of V are therefore $\mu = 1$, when $\bar{\mu} > 1$, and the limit $\mu \rightarrow \infty$. The function $\ln(1 + \tau) - \tau(1 - e^{-1})$ is strictly concave in τ , zero at $\tau = 0$ with slope $e^{-1} > 0$, and tends to $-\infty$ as $\tau \rightarrow \infty$, so it has a unique positive zero $\hat{\tau}$ and is positive exactly on $(0, \hat{\tau})$. Comparing the candidates,

$$V^M(1) \geq \bar{V} \iff e^{-\tau/e} \leq (1 + \tau)e^{-\tau} \iff \tau(1 - e^{-1}) \leq \ln(1 + \tau) \iff \tau \leq \hat{\tau}.$$

If $\tau > \hat{\tau}$, then $V < \bar{V}$ everywhere, as $V^M \leq V^M(1) < \bar{V}$ on the mixed region and $V^C < \bar{V}$ on the pure region, so V approaches its supremum only as $\mu \rightarrow \infty$. If $\tau \leq \hat{\tau}$, then $\bar{\mu} > 1$: at $\mu = 1$, $\varepsilon = \tau(1 - e^{-1}) < \tau$, so $V^C(1) < \bar{V} \leq V^M(1)$, which by Lemma C.4 places $\mu = 1$ below $\bar{\mu}$. Hence $V(1) = V^M(1) \geq \bar{V} > V(\mu)$ for $\mu \geq \bar{\mu}$, and $V(\mu) = V^M(\mu) < V^M(1)$ for $\mu < \bar{\mu}$ with $\mu \neq 1$: V is maximized only at $\mu = 1$. ■

References

- BAYE, M. R., J. MORGAN, AND P. SCHOLTEN (2004): “Price dispersion in the small and in the large: Evidence from an internet price comparison site,” *The Journal of Industrial Economics*, 52(4), 463–496.
- BETHUNE, Z., M. CHOI, AND R. WRIGHT (2020): “Frictional goods markets: Theory and applications,” *The Review of Economic Studies*, 87(2), 691–720.
- BROECKER, T. (1990): “Credit-worthiness tests and interbank competition,” *Econometrica*, 58(2), 429–452.
- BURDETT, K., AND K. L. JUDD (1983): “Equilibrium price dispersion,” *Econometrica*, 51(4), 955–969.
- BURDETT, K., S. SHI, AND R. WRIGHT (2001): “Pricing and matching with frictions,” *Journal of Political Economy*, 109(5), 1060–1085.
- BUTTERS, G. R. (1977): “Equilibrium distributions of sales and advertising prices,” *The Review of Economic Studies*, 44(3), 465–491.
- CAMPO, K., E. GIJSBRECHTS, AND P. NISOL (2003): “The impact of retailer stock-

- outs on whether, how much, and what to buy,” *International Journal of Research in Marketing*, 20(3), 273–286.
- COHEN, A., M. KARGAR, B. LESTER, AND P.-O. WEILL (2024): “Inventory, market making, and liquidity in OTC markets,” *Journal of Economic Theory*, 222, 105917.
- ELSBY, M. W., R. MICHAELS, AND D. RATNER (2015): “The Beveridge curve: A survey,” *Journal of Economic Literature*, 53(3), 571–630.
- GALENIANOS, M., AND P. KIRCHER (2012): “On the game-theoretic foundations of competitive search equilibrium,” *International Economic Review*, 53(1), 1–21.
- GOMIS-PORQUERAS, P., B. JULIEN, AND C. WANG (2017): “Strategic advertising and directed search,” *International Economic Review*, 58(3), 783–806.
- HOSIOS, A. J. (1990): “On the efficiency of matching and related models of search and unemployment,” *The Review of Economic Studies*, 57(2), 279–298.
- HUANG, C. (2016): “Price dispersion in a directed search model,” *Canadian Journal of Economics*, 49(3), 1208–1223.
- KAPLAN, G., AND G. MENZIO (2015): “The morphology of price dispersion,” *International Economic Review*, 56(4), 1165–1206.
- KAPLAN, G., G. MENZIO, L. RUDANKO, AND N. TRACHTER (2019): “Relative price dispersion: Evidence and theory,” *American Economic Journal: Microeconomics*, 11(3), 68–124.
- LAGOS, R., G. ROCHETEAU, AND R. WRIGHT (2017): “Liquidity: A new monetarist perspective,” *Journal of Economic Literature*, 55(2), 371–440.
- LESTER, B. (2011): “Information and prices with capacity constraints,” *American Economic Review*, 101(4), 1591–1600.
- MENZIO, G. (2024): “Search theory of imperfect competition with decreasing returns to scale,” *Journal of Economic Theory*, 218, 105827.
- MONDERER, D., AND L. S. SHAPLEY (1996): “Potential games,” *Games and Economic Behavior*, 14(1), 124–143.
- MONTGOMERY, J. D. (1991): “Equilibrium wage dispersion and interindustry wage differentials,” *The Quarterly Journal of Economics*, 106(1), 163–179.
- NARASIMHAN, C. (1988): “Competitive promotional strategies,” *Journal of Business*, 61(4), 427–449.
- PETERS, M. (1984): “Bertrand equilibrium with capacity constraints and restricted

- mobility,” *Econometrica*, 52(5), 1117–1127.
- (1991): “Ex ante price offers in matching games non-steady states,” *Econometrica*, 59(5), 1425–1454.
- (2000): “Limits of exact equilibria for capacity constrained sellers with costly search,” *Journal of Economic Theory*, 95(2), 139–168.
- PRATT, J. W., D. A. WISE, AND R. ZECKHAUSER (1979): “Price differences in almost competitive markets,” *The Quarterly Journal of Economics*, 93(2), 189–211.
- REKKAS, M., L. JIANG, R. WRIGHT, AND Y. ZHU (2025): “Price dispersion and price stickiness in a competitive search model of housing markets,” *Economic Theory*, pp. 1–36.
- ROGERSON, R., R. SHIMER, AND R. WRIGHT (2005): “Search-theoretic models of the labor market: A survey,” *Journal of Economic Literature*, 43(4), 959–988.
- SHILONY, Y. (1977): “Mixed pricing in oligopoly,” *Journal of Economic Theory*, 14(2), 373–388.
- SORENSEN, A. T. (2000): “Equilibrium price dispersion in retail markets for prescription drugs,” *Journal of Political Economy*, 108(4), 833–850.
- STIGLER, G. J. (1961): “The economics of information,” *Journal of Political Economy*, 69(3), 213–225.
- VARIAN, H. R. (1980): “A model of sales,” *The American Economic Review*, 70(4), 651–659.
- WATANABE, M. (2010): “A model of merchants,” *Journal of Economic Theory*, 145(5), 1865–1889.
- WRIGHT, R., P. KIRCHER, B. JULIEN, AND V. GUERRIERI (2021): “Directed search and competitive search equilibrium: A guided tour,” *Journal of Economic Literature*, 59(1), 90–148.